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Some Structural Properties of Project Digraphs*

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Project digraphs (which are finite, directed, simple, acyclic, weakly connected graphs with exactly one transmitter and one receiver point) are of importance, e.g., when dealing with scheduling problems. A decomposition of such digraphs by means of systems of subproject digraphs is proposed and some structural properties mainly in view to conditions for routability possibilities of paths are described which are helpful, e.g., in situations when for projcet planning models stochastic variation of data impose problems for project duration time estimation.

1. INTRODUCTION

It is known that the problem of finding a minimum decomposition of a finite partially ordered set into chains, see Dilworth [1] (or antichains, see Mirsky [6]) has important applications e.g. for scheduling problems as discussed with consideration to network flow methods in Fulkerson [2]. Adequate models can be described (as done in [2]) in terms of so-called project digraphs. In this context it should be mentioned that representation of a partially ordered set by the *arc set* of a corresponding digraph (the (strict) order corresponds to the transitive closure of a relation r which for two arcs x_1, x_2 puts $x_1 r x_2$ iff endpoint of x_1 equals starting point of x_2) can impose the use of dummy arcs; in other words, if the partially ordered set is not graphically representable in the above mentioned way there always exists an enlargement of the set together with an enlarged order (where the restriction of which to the underlying original set coincides with the given order) that will do, but how to construct such a graphtheoretical representation efficiently is a problem of more practical importance and not considered here.

A decomposition scheme for such project digraphs by means of proper subproject digraph systems and some structural properties of such decompositions are given in this paper. These properties mainly deal with conditions for the routability of paths which is of interest e.g. for the consideration of project planning models in situations when stochastic variation of data impose problems for the estimation of expected project duration time, see [3].

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2. Project Digraphs

For ease of description the notation of Harary [4] and of Harary, Norman and Cartwright [5] is mostly adopted.

A project digraph $D_{tr} = (V, X, f)$ is a finite, directed, simple, acyclic, weakly connected graph where $V \neq \emptyset$ denotes the set of points, X the set of arcs and $f = (f^1, f^2)$ with $f^i : X \to V$, $i = 1, 2$, the injective incidence mapping where $f^1(x)$ resp. $f^2(x)$ gives the starting resp. end point of $x \in X$ (an additional notation, sometimes used for arcs x is $f^1(x)f^2(x)$) with exactly two specified points $t, r \in V$, called transmitter, receiver with $\{x \mid x \in X, f^2(x) = t\} = \emptyset$, $\{x \mid x \in X, f^1(x) = r\} = \emptyset$*.

For project digraphs there exists at least one bijective mapping called, (ascending) level-assignment $1 : V \to \{0, 1, \ldots, m\}$, $m := |V| - 1$, with $x \in X \Rightarrow 1(f^1(x)) < 1(f^2(x))$ (and $1(t) = 0$, $1(r) = m$), see e.g. [5, p. 267]. For notational convenience the identification $V := \{0, 1, \ldots, m\}$ is made, and it is assumed that the points have been topologically ordered according to a level-assignment.

A subgraph $D_{ij} = (V_{ij}, X_{ij}, f_{ij})(i, j \in V_{ij} \subset V, X_{ij} \subset X, f_{ij} = f/X_{ij})$ is called subproject digraph, if D_{ij} is a project digraph with transmitter i and receiver j.

The incidence mapping is mostly omitted, to put it more briefly, D_{ij}, solely, is written. In this case, $V(D_{ij})$ resp. $X(D_{ij})$ is used to denote the point resp. arc set of D_{ij}.

Let $S \subset \{(i, j) \mid i, j \in V, i \leqslant j\}$ be the set of points for which subproject digraphs exist. Subproject digraphs which are maximal with respect to $(i, j) \in S$ are denoted by $\tilde{D}_{ij} = (\tilde{V}_{ij}, \tilde{X}_{ij})$, hence, in this notation D_{tr} is replaced by

$$\tilde{D}_{0m} = (V, X) \qquad (= (\tilde{V}_{0m}, \tilde{X}_{0m}))$$

which is used from now on.

A path P_{ij} (an alternating sequence of points and arcs of the form $i = i_0, i_0i_1, i_1, \ldots, i_{n-1}, i_{n-1}i_n, i_n = j$, where the sequence of points consists of distinct elements only) is a special subproject digraph. $(P_{ij})_k$ resp. $(P_{ij})^k$, $k \in V(P_{ij})$, denotes the subpath of P_{ij} from i to k resp. k to j. For a subproject digraph D_{ij} the set of paths P_{ij} belonging to D_{ij} is given by $P(D_{ij})$.

Obviously, one has

Lemma 1. *If $D_{ij} = (V_{ij}, X_{ij})$ is a subproject digraph with transmitter i and receiver j, then i is a source†, j is a sink of D_{ij}.*

* Acyclic digraphs possess at least one transmitter and one receiver point, see [4, p. 200].

† A source (sink) is a point basis (point contrabasis) consisting of a single element see [5, p. 98].

3. Subproject Digraph Systems

For $v \in V$, $v \geqslant 1$, consider systems of subproject digraphs of the form $\vartheta_v = \{D_{iv} \mid D_{iv} \text{ is subproject digraph, } i < v\}$.

Let be $B(\vartheta_v) = \{i \mid i \in V,\ D_{iv} \in \vartheta_v\}$.

ϑ_v is called a proper subproject digraph system if*

$$D^1_{i_1 v},\ D^2_{i_2 v} \in \vartheta_v \Rightarrow D^1_{i_1 v} \cap D^2_{i_2 v} = \begin{cases} (\{i, v\}, \emptyset) & i_1 = i_2 = i \\ (\{v\}, \emptyset) & \text{otherwise} \end{cases} \tag{1}$$

$$\forall\ P_{0v} \subset \tilde{D}_{0m}\ \exists\ D_{iv} \in \vartheta_v : P_{0v} = (P_{0v})_i \cup (P_{0v})^i \text{ with } \begin{matrix} (P_{0v})^i \in P(D_{iv}) \\ (P_{0v})_i \cap \vartheta_v \subset (B(\vartheta_v), \emptyset) \end{matrix} \tag{2}$$

Choosing subproject digraph systems according to (1), (2) establishes some conditions which have to be explained next.

First of all, such proper systems do always exist, e.g. $\vartheta^1_v = \tilde{D}_{0v}$ and $\vartheta^2_v = \{x \mid x \in X, f^2(x) = v\}$ are proper.

Among the properties for proper systems one immediately recognizes.

Lemma 2. *If $\vartheta_v = \{D_{iv}\}$ is a proper system, then*

$$\vartheta^1_v \supset \vartheta_v \supset \vartheta^2_v.$$

Proof. It suffices to check the arc-sets.

Let be $x \in X(\vartheta^2_v)$ but $x \notin X(\vartheta_v)$ then $x \notin X((P_{0v})^i)$ for all P_{0v} and all $D_{iv} \in \vartheta$ because of (2), thus because $\tilde{D}_{0m}$ is finite and acyclic there exists $\hat{x} \in Q(x) := \{x' \mid x' \in X, \exists\ P_{f^1(x') f^1(x)} \subset \tilde{D}_{0m}\}$ with $f^1(\hat{x}) = \min_{x' \in Q(x)} f^1(x') > 0$, but $f^1(\hat{x})$ is a transmitter of $\tilde{D}_{0m}$ which is a contradiction to Lemma 1 applied to $\tilde{D}_{0m}$.

Now, knowing that each $x \in X$ must be traceable to source O (by a path $P_{0f^1(x)}$) and assuming $x \in X(_v)$ (which assures the existence of a path $P_{f^2(x)v}$ by similar arguments) but $x \notin X(\vartheta^1_v)$ yields

$$\bar{P}_{0v} := P_{0f^1(x)} \cup \{x\} \cup P_{f^2(x)v} \notin P(\tilde{D}_{0v})$$

which contradicts the maximality of $\tilde{D}_{0v}$. ∎

There are close connections between the choice of a proper system and conditions for the routability of paths which can be seen from

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* For digraphs $D_1 = (V_1, X_1)$, $D_2 = (V_2, X_2)$ the following notation is used:

$$D_1 \subset D_2 \text{ iff } V_1 \subset V_2 \text{ and } X_1 \subset X_2,\ D_1 \overset{\cup}{\cap} D_2 := (V_1 \overset{\cup}{\cap} V_2,\ X_1 \overset{\cup}{\cap} Y_2).$$

If D_1, D_2 are as well point as arc disjoint only one symbol $\emptyset$ is written. Sometimes it is convenient to regard ϑ_v as union of its subproject digraphs.

THEOREM 1. *If* $\vartheta_v = \{D_{iv}\}$ *is a proper system, then*

(*i*) $\forall P_{jk} \subset \tilde{D}_{0m}$
$x \in X(P_{jk}) \wedge f^2(x) \in V(\vartheta_v) \setminus B(\vartheta_v) \Rightarrow \exists D_{i_0v} \in \vartheta_v$ *with*
$x \in X(D_{i_0v})$

(*ii*) $\forall P_{jk} \subset \tilde{D}_{0m}$
$x \in X(P_{jk}) \wedge f^1(x) \in V(\vartheta_v) \setminus B(\vartheta_v) \Rightarrow$ *either* $\exists D_{i_0v} \in \vartheta_v$ *with*
$x \in X(D_{i_0v})$ or $(P_{jk})^{f^2(x)} \cap \vartheta_v = \emptyset$.

To prove Theorem 1 *it is convenient to develop a preliminary result.*

LEMMA 3. *If* $\vartheta_v = \{D_{iv}\}$ *is a proper system, then*

$$\forall P_{0p} \subset \tilde{D}_{0m}(p \in V(\vartheta_v)) \exists D_{i_0v} \in \vartheta_v : P_{0p} = (P_{0p})_{i_0} \cup (P_{0p})^{i_0}$$

and $$(P_{0p})_{i_0} \cap \vartheta_v \subset (B(\vartheta_v), \emptyset),\ (P_{0p})^{i_0} \subset D_{i_0v}.$$

Proof. For $p = v$ one gets (2), for $p \in B(\vartheta_v)$ considering the trivial decomposition $P_{0p} = (P_{0p})_p \cup (P_{0p})^p$ which allows to choose $i_0 = p$ P_{0p} can be regarded as initial part $(P_{0v})_{i_0}$ of a path P_{0v} with $(P_{0v})^{i_0} \in P(D_{i_0v})$ for an appropriate $D_{i_0v} \in \vartheta_v$, thus $P_{0p} \cap \vartheta_v \subset (B(\vartheta_v), \emptyset)$. Now, let be $p \in V(D_{i_0v})$ with $i_0 < p < v$. From lemma 1 applied to D_{i_0v} there exists $P_{pv} \subset D_{i_0v}$, thus from (2) for $\bar{P}_{0v} := P_{0p} \cup P_{pv}$ there exists $D_{i_1v} \in \vartheta_v$ and a decomposition $(\bar{P}_{0v})_{i_1} \cup (\bar{P}_{0v})^{i_1}$ with $(\bar{P}_{0v})_{i_1} \cap \vartheta_v \subset (B(\vartheta_v), \emptyset)$, thus $p \notin (\bar{P}_{0v})_{i_1}$ but the other possibility yields $p \in V(D_{i_0v} \cap D_{i_1v}) \setminus \{i_0, v\})$ (and from (1) different subproject digraphs D_{i_0v}, D_{i_1v} must be point-disjoint except for v, and eventually $i_0 = i_1$), thus $D_{i_1v} = D_{i_0v}$ and $(\bar{P}_{0v})^{i_1} = ((\bar{P}_{0v})^{i_1})_p \cup (\bar{P}_{0v})^p$ with $((\bar{P}_{0v})^{i_1})_p = (P_{0p})^{i_0} \subset D_{i_0v}$, $(\bar{P}_{0v})_{i_1} = (P_{0p})_{i_0}$ with $(P_{0p})_{i_0} \cap \vartheta_v \subset (B(\vartheta_v), \emptyset)$. ∎

Proof of Theorem 1. (i) Construct a path $P_{0f^2(x)}$ with $x \in X(P_{0f^2(x)})$ which exists because of Lemma 1 and apply Lemma 3.

(ii) Assume $x \cap \vartheta_v \subset (\{f^1(x), f^2(x)\}, \emptyset)$ and $M := (P_{jk})^{f^2(x)} \cap \vartheta_v \neq \emptyset$. Choosing $v_0 \in V(M)$ minimal yields $v_0 \in B(\vartheta_v)$ because of (i). Now, construct $P_{0f^1(x)}$ which exists because of Lemma 1 but for $\bar{P}_{0v_0} := P_{0f^1(x)} \cup \{x\} \cup ((P_{jk})^{f^2(x)})_{v_0}$ Lemma 3 would not hold, thus $M = \emptyset$. ∎

Up to now, besides Lemma 2 no information about the relations between different subproject digraph systems was obtained.

One can prove.

THEOREM 2. *Let be* $\vartheta_v = \{D_{iv}\}$, $\vartheta'_v = \{D'_{jv}\}$ *be two proper systems with* $X(\vartheta_v) \supset X(\vartheta'_v)$. *Then there exists a proper system* $\vartheta''_v = \{D''_{qv}\}$ *with*

(*i*) $X(\vartheta''_v) = X(\vartheta_v)$.

(*ii*) $\forall D'_{jv} \in \vartheta'_v \exists D''_{qv} \in \vartheta''_v : D'_{jv} \subset D''_{qv}$.

Proof. If one can choose $\vartheta''_v = \vartheta_v$ (if ϑ_v satisfies (ii)) nothing remains to be shown, otherwise exist $D'_{j_0v} \in \vartheta'_v$ and, at least, $D^1_{i_1v}, D^2_{i_2v} \in \vartheta_v$ (see lemma 2) with

$$D^1_{i_1v} \cap D'_{j_0v} \supsetneqq (\{v\}, \phi), \qquad D^2_{i_2v} \cap D'_{j_0v} \supsetneqq (\{v\}, \phi). \tag{3}$$

But (3) forces $i_1 = i_2 = j_0$.

$i_1 = i_2 = j_0$ is equivalent to $j_0 \in V(D^1_{i_1v} \cap D^2_{i_2v})$, and $j_0 \in V(D^1_{i_1v} \cap D^2_{i_2v})$ must be valid, because otherwise $j_0 \notin V(D^{\hat{1}}_{i_{\hat{1}}v})$ together with (3) for $1 = \hat{1}$ means there exists $v_0 \in V(D^{\hat{1}}_{i_{\hat{1}}v} \cap D'_{j_0v})$ with $j_0 < v_0 < v$ and $P_{j_0v_0} \subset D'_{j_0v})$ (from lemma 1). Now applying theorem 1 (with respect to ϑ_v) yields $i_{\hat{1}} \in V(P_{j_0v_0}) \subset V(D'_{j_0v})$ with $j_0 < i_{\hat{1}}$ and $(P_{j_0v_0})_{i_{\hat{1}}} \subset D'_{j_0v} \subset \vartheta'_v \subset \vartheta_v$ and there must exist $D_{iv} \in \vartheta_v$ with $i \leqslant j_0 < i_{\hat{1}} \leqslant v_0 < v$ and $(P_{j_0v_0})_{i_{\hat{1}}} \subset D_{iv}$ but $D_{iv} \cap D^{\hat{1}}_{i_{\hat{1}}v} \supset (\{i_1, v), \emptyset)$ for $i \neq i_{\hat{1}}$, a contradiction to (1).

$i_1 = i_2 = j_0 = i^*$ being valid, construct

$$\vartheta^*_v = \vartheta(_v \setminus \{D^1_{i^*v}, D^2_{i^*v}\}) \cup D^*_{i^*v} \text{ with } D^*_{i^*v} := D^1_{i^*v} \cup D^2_{i^*v}$$

ϑ^*_v can easily be seen to be a proper system satisfying (i) but with a reduced number of subproject digraphs.

If (ii) is not yet satisfied the above described procedure is repeated. ∎

For $\vartheta'_v, \vartheta''_v$ fulfilling the condition of Theorem 2 the abbreviated notation $\vartheta'_v \Subset \vartheta''_v$ is used.

The following corollary gives a hint how to construct a proper system that covers a given one according to $\Subset$.

COROLLARY. *Let* $\vartheta'_v = \{D'_{jv}\}$, $\vartheta''_v = \{D''_{qv}\}$ *be two proper systems with* $\vartheta'_v \Subset \vartheta''_v$ *then*

$$\forall D''_{qv} \in \vartheta''_v \quad \forall j \in B(\vartheta'_v) \cap V(D''_{qv}) - \{q\} : D''_{qv} \supset \tilde{D}_{qj} \cup D'_{jv}$$

Proof is now obvious from theorems 1, 2 and omitted.

4. Conclusion

Decompositions of project digraphs according to proper subproject digraph systems may be of interest in situations when the structure of the underlying digraph is too voluminous to be handled directly. The special decomposition scheme described here fits, e.g. for scheduling problems when one has to determine characteristic values for the points and arcs of $\tilde{D}_{0m}$ taking into account the information given by the subproject digraph systems.

Surely, more comprehensive systems for which a representation satisfying $\Subset$ should be used may possess more informations than a given one and can yield improved characteristic values (in this context for $v \in V$, $v \geqslant 1$, $\tilde{D}_{0v}$ would be the best one to be chosen) but less comprehensive systems may be easier to handle. For stochastic project planning models the concept, presented here, can be used by including recursive arguments to find estimators for expected project duration times which improve those known from the literature, see [3].

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