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<u>Remarks on stochastic aspects on graphs</u>

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1. Introduction

For the description and solution of many problems in the area of operations research for which graph models have turned out to be an appropriate tool if problem data are fixed and known to enlarge the possibilities of application it is desirable to find corresponding formulations if variation of data has to be taken into consideration. With regard to this view-point problems including stochastic variation of data are of special interest and some efforts in combining stochastic and graphtheoretical aspects are cited in the references omitting, however, the large number of contributions using tools from reliability theory and stochastic programming (except [3] [15]). Of course, maximum flow and shortest path network formulations belong to the problems for which stochastic generalizations were considered (see e.g. [6], [9], [25]), however, stochastic versions of project planning models seem to be mostly dealt with in this context. Defining a project to be a finite set of activities $A = \{a_1, \ldots, a_n\} \neq \emptyset$ with an irreflexive, asymmetric, transitive ordering relation $\mathcal{O} \subset A \times A$ (which allows representation as an adjacency-relation on the arc-set of a graph at least after introducing dummy activities, see e.g [1], [20]) and nonnegative activity durations the problem is to determine the project duration which is yielded by maximizing over the sums of the durations of those activities which form maximal chains (with respect to $\mathcal{O}$). Assuming stochastic activity durations an intuitive idea is to replace the stochastic variables by their expected values and solve the resulting deterministic problem (using CPM), thus providing a rough estimate for the project duration mean.
For this PERT-approach [18] several improvements have been suggested (see [4], [5], [10], [16], [22]). This paper gives a description of a class of estimation possibilities containig some estimators known from the literature as special cases.
To get information about the distribution of the project duration one can also use simulation techniques (see e.g. [2], [12], [17], [23]) or try to reduce or decompose the project structure (see e.g. [2], [13] [14], [19]). Some decompositions of the project and properties of sub-

project systems are also given in this paper. A further generalization considering additional stochastic aspects influencing the project structure is not handled in this context (see e.g. [7], [12], [21] for an introduction to those problems).

2. Project-graphs

Let

(2.1) $\overset{\frown}{N}_{om} = (P,K,\varphi,X)$

denote a (stochastic) project-graph[1] (with deterministic structure) which means that (P,K,φ) is a finite, directed, simple, acyclic, (weak) connected graph where P describes the set of nodes, K the set of arcs and $\varphi = (\varphi^1,\varphi^2)$ with $\varphi^i: K \longrightarrow P$, $i=1,2$, the injective incidence mapping of the graph where $\varphi^1(k)$ resp. $\varphi^2(k)$ gives the starting-resp. end-point of $k \in K$. Additionally, (P,K,φ) posesses two nodes $s,t \in P$, called source, sink with $\{k \in K \mid \varphi^2(k) = s\} = \emptyset$, $\{k \in K \mid \varphi^1(k) = t\} = \emptyset$.
For such graphs a bijective mapping (called topological order of the nodes) $\sigma : P \longrightarrow \{0,1,\dots,m\}$, $m := |P| - 1$, satisfying $k \in K \Rightarrow \sigma(\varphi^1(k)) < \sigma(\varphi^2(k))$ (with $\sigma(s) = 0$, $\sigma(t) = m$) exists, and for ease of description, it is assumed that $P := \{0,1,\dots,m\}$ is topologically ordered in the mentioned way.
For $\widetilde{K} \subset K$ let $X_{\widetilde{K}}$ denote the vector of random variables $X_k, k \in \widetilde{K}$ defined on a probability space $(\Omega,\mathfrak{S},Pr)$ (sometimes the subscript $\widetilde{K}$ is omitted) describing the arc passage times. $X_k \in \mathcal{L}_1$, $k \in K$, and, if $\bigcup_{i=1}^{\varsigma} K_i$ is an appropriate partition of K, independence for $X_{K_1},\dots,X_{K_\varsigma}$, and the knowledge of the joint distribution of X_{K_i}, $i \in \{1,\dots,\varsigma\}$ is assumed (For ease of description often the special case $|K_i| = 1$ is choosen.).
For $p \in P$ one defines

(2.2) $K(p) := \{k \in K \mid \varphi^2(k) = p\}$, $B(p) := \{j \in P \mid \varphi^1(k) = j, k \in K(p)\}$

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1) $\overset{\frown}{N}_{om}$ can be considered as graphtheoretical representation of a project planning model where no stochastic aspects influence the project structure.

from which one obtains the following subgraph of $\widetilde{N}_{om}$

(2.3) $$G_p = (P_p, K_p, \varphi_p)$$

with $P_p := \{0,1\dots,p\}$, $K_p := \bigcup_{\mu=0}^{p} K(\mu)$ (as a partition of $\varphi^{-1}(P_p \times P_p \cap \varphi(K)))$, $\varphi_p = \varphi/K_p$.

Of special interest are subproject-graphs for $i,j \in P$, $i \leq j$, which (if existent) are denoted by

(2.4) $$N_{ij} = (P_{ij}, K_{ij}, \varphi_{ij}, X_{K_{ij}})$$

where $P_{ij} \subset \{i,i+1,\dots,j-1,j\}$ ($i,j \in P_{ij}$ is source, sink of N_{ij}), $K_{ij} \subset \varphi^{-1}(P_{ij} \times P_{ij} \cap \varphi(K))$ have appropriately to be choosen to fit the project-graph conditions, $\varphi_{ij} = \varphi/K_{ij}$.
$\widetilde{N}_{op} = (\widetilde{P}_{op}, \widetilde{K}_{op}, \widetilde{\varphi}_{op}, X_{\widetilde{K}_{op}})$ denotes the subproject-graph which is maximal in G_p.
If W_{ij} describes a route from i to j , $i,j \in P$ (W_{ij} is a special subproject-graph), $K(W_{ij})$ the arc set of W_{ij} and $\mathfrak{W}(N_{ij})$ the set of routes W_{ij} belonging to N_{ij} , one can define

(2.5) $$L(N_{ij}) := \max_{\mathfrak{W}(N_{ij})} \sum_{k \in K(W_{ij})} proj_k$$

which gives the N_{ij}-project passage time. Of special interest are

(2.6) $$L_p := L(\widetilde{N}_{op}) \quad , p \in P$$

(with $L_o := 0$, $L_m := L(\widetilde{N}_{om})$).

3. Subproject-graph systems

For $p \in P$, $p \geq 1$, consider systems of subproject-graphs of the form

(3.1) $$\mathfrak{N}_p := \{ N_{ip} | N_{ip}, i<p, \text{subproject-graph}\} \text{ with } B(\mathfrak{N}_p) := \{i \in P | N_{ip} \in \mathfrak{N}_p\}$$

$\mathfrak{N}_p$ is called proper subproject-graph system if[2)]

[2)] For graphs $G_1 = (P_1, K_1)$, $G_2 = (P_2, K_2)$ omitting the incidence mappings φ_1, φ_2 (and stochastic vectors X_{K_1}, X_{K_2}) the following notation is used
$G_1 \subset G_2$ iff $P_1 \subset P_2$ and $K_1 \subset K_2$ and $G_1 \cup\!\!\cap\, G_2 = (P_1 \cup\!\!\cap\, P_2, K_1 \cup\!\!\cap\, K_2)$ where, if no ambiguity can arise, only the relevant set in brackets is written.

(3.2) $\quad N^1_{i_1p}\ ,\ N^2_{i_2p} \in \mathfrak{N}_p \Rightarrow N^1_{i_1p} \cap N^2_{i_2p} = \begin{cases} \{i,p\}, i_1=i_2=i \\ \{p\}\ , \text{otherwise} \end{cases}$

(3.3) $\quad \forall\, W_{op} \exists\, N_{ip} \in \mathfrak{N}_p : W_{op} = W_{oi} \cup W_{ip}$ with $W_{ip} \subset N_{ip}$

There always exist proper (subproject-graph) systems , e.g. $\mathfrak{N}_p = \{\tilde{N}_{op}\}$ is always proper.
Among the properties for proper systems one has

(3.4) Theorem:

Let $\mathfrak{N}_p = \{N_{ip}\}$, $\mathfrak{N}'_p = \{N'_{jp}\}$ be two proper systems with $\bigcup_{\mathfrak{N}_p} K_{ip} \supset \bigcup_{\mathfrak{N}'_p} K'_{jp}$. Then there exists a proper system $\mathfrak{N}''_p = \{N''_{qp}\}$ with

i) $\bigcup_{\mathfrak{N}''_p} K''_{qp} = \bigcup_{\mathfrak{N}_p} K_{ip}$ and

ii) $\forall\, N'_{jp} \in \mathfrak{N}'_p\ \exists\ N''_{qp} \in \mathfrak{N}''_p : N'_{jp} \subset N''_{qp}$

For a proof see [11].
let be

(3.5) $\quad \mathfrak{N}'_p \in \mathfrak{N}''_p$

the abreviated notation for the situation described in (3.4).
The next theorem shows why proper subproject-graph systems are useful.

(3.6) Theorem:

If $\mathfrak{N}_q = \{N_{iq}\}$ is a proper system, $q \in P \setminus \{0\}$, one has

$$L_q = \max_{B(\mathfrak{N}_q)} \{ L_i + L(N_{iq}) \}$$

For a proof one has to consider that for $\omega \in \Omega$ and $i_1 \in B(\mathfrak{N}_q)$ there exist $\hat{W}_{oi_1} \in \mathcal{W}(\tilde{N}_{oi_1})$, $\hat{W}_{i_1q} \in \mathcal{W}(N_{i_1q})$ which maximize (2.6) (for $p=i_1$) and (2.5) (for $i=i_1, j=q$) for the realization $X(\omega)$, but $\hat{W}_{oi_1} \cup \hat{W}_{i_1q} \in \mathcal{W}(\tilde{N}_{oq})$, thus

$$L_{i_1}(X(\omega)) + L(N_{i_1q})(X(\omega)) \leq L_q(X(\omega))$$

On the other side if $\hat{W}_{oq} \in \mathcal{W}(\tilde{N}_{oq})$ maximizes (2.6) (for $p=q$) for $X(\omega)$ because of (3.3) there exists $N_{i_2q} \in \mathfrak{N}_q$ such that

$\widehat{W}_{oq} = W_{oi_2} \cup W_{i_2q}$ with $W_{i_2q} \in \mathcal{W}(N_{i_2q})$ and, of course,
$W_{oi_2} \in \mathcal{W}(\widetilde{N}_{oi_2})$, thus

$$L_q(X(\omega)) = \sum_{k \in K(W_{oi_2})} \mathrm{proj}_k(X(\omega)) + \sum_{k \in K(W_{i_2q})} \mathrm{proj}_k(X(\omega))$$
$$\leq L_{i_2}(X(\omega)) + L(N_{i_2q})(X(\omega)) .$$

Now, maximizing with respect to $B(\pi_q)$ gives the theorem.

4. Project passage time estimation

For ease description X_k , $k \in K$, are assumed independent for the present. For $Z \subset K$ let E_Z denote the integration with respect to X_Z , E (without subscript) the expectation.
Then, for $p \in P$, $p \geq 1$, if S_j are known optimistic estimators for the $\widetilde{N}_{oj}$ -project passage times which means

(4.1) $$S_j \leq E(L_j) \quad , j \in \{0,1,\ldots,p-1\} \text{ , with } S_o := 0$$

one can define

(4.2) $$S(\pi_p, Z_p, L_p) := E_{Z_p} \max_{B(\pi_p)} \{ S_i + E_{\overline{Z}_p}(L(N_{ip})) \}$$

where $\pi_p = \{N_{ip}\}$ is a proper system , $\{Z_p, \overline{Z}_p\}$ a partition of K.
With the restriction to partitions which satisfy

(4.3) $$Z_p \subset \bigcup_{\pi_p} K_{ip}$$

one gets

(4.4) Theorem:

(i) Let $\pi_p = \{N_{ip}\}$ be a proper system, $\{Z_p, \overline{Z}_p\}$ a partition of K satisfying (4.3) then

$$E(L_p) \geq S(\pi_p, Z_p, L_p)$$

(ii) Let $\{Z_p^i, \overline{Z_p^i}\}$ be a partition of K , i=1,2 ,and $Z_p^1 \supset Z_p^2$ then

$$S(\pi_p, Z_p^1, L_p) \geq S(\pi_p, Z_p^2, L_p)$$

From (4.4) one can see that the best to do with respect to $\mathcal{N}_p$ is to choose the partition $\{Z_p^*, \overline{Z_p^*}\}$ with $Z_p^* = \bigcup_{\mathcal{N}_p} K_{ip}$, for which $S(\mathcal{N}_p, Z_p^*, L_p)$, additionally, has the simplified form

$$(4.5) \qquad S(\mathcal{N}_p, Z_p^*, L_p) = E_{Z_p^*} \max_{B(\mathcal{N}_p)} \{ S_i + L(N_{ip}) \}$$

The next question is, wether there is a possibility to improve the estimator by changing from one subproject-graph system to another. One can show

(4.6) Theorem:

If $\mathcal{N}_p' = \{N_{jp}'\}$ with $Z_p^{*'} = \bigcup_{\mathcal{N}_p'} K_{jp}'$, $\mathcal{N}_p'' = \{N_{qp}''\}$ with $Z_p^{*''} = \bigcup_{\mathcal{N}_p''} K_{qp}''$ are two proper systems with $\mathcal{N}_p' \in \mathcal{N}_p''$ then

$$S(\mathcal{N}_p'', Z_p^{*''}, L_p) \geq S(\mathcal{N}_p', Z_p^{*'}, L_p)$$

if $E(\tilde{N}_{qj}) \geq S_j - S_q$ for all $q \in B(\mathcal{N}_p'')$, $j \in B(\mathcal{N}_p') \cap N_{qp}''$ (where $\tilde{N}_{qj}$ denotes the maximal subproject-graph with respect $q, j \in P$)

For a proof of (4.4),(4.6) see [11].

From this representation one can get estimators known from the literature as special cases.
For instance, choosing S_j^1 as PERT-estimators for the $\tilde{N}_{oj}$-project passage times , $j \in \{0,1,\dots,p-1\}$, $\mathcal{N}_p^1 = K(p)$ (see (2.2)) from which one gets $B(\mathcal{N}_p^1) = B(p)$, $Z_p^1 = \emptyset$, gives $S_p^1 = S(K(p), \emptyset, L_p) = \max_{B(p)} \{ S_i^1 + E(X_{\varphi^{-1}(i,p)}) \}$,the PERT-estimation.

Taking $\mathcal{N}_p^2 = \mathcal{N}_p^1$, but $Z_p^2 = K(p)$ and $S_j^2 \geqslant S_j^1$ (where S_j^2 satisfy (4.1),see [10]) gives

$$S_p^2 = S(K(p), K(p), L_p) = E_{K(p)} \max_{B(p)} \{ S_i^2 + X_{\varphi^{-1}(i,p)} \}$$

and from (4.4) it follows $S_p^2 \geq S_p^1$.

Choosing $\mathcal{N}_p^3 = \{ W_{ip} \mid W_{ip}$ is route from i to p $, i, p \in P\}$ in a proper way (satisfying (3.2),(3.3)) and $S_j^3 \geqslant S_j^2$ for $j \in B(\mathcal{N}_j^3)$ (where S_j^3 satisfy (4.1),see [22]) and $Z_p^3 = \bigcup_{\mathcal{N}_p^3} K(W_{ip})$ gives

$$S_p^3 = S(\{W_{ip}\}, Z_p^3, L_p) = E_{Z_p^3} \max_{B(\{W_{ip}\})} \{S_i^3 + L(W_{ip})\} \geq S_p^2$$

because $\mathfrak{N}_p^3, \mathfrak{N}_p^2$ fulfill (3.5) and (4.6).
At last, choosing $\mathfrak{N}_p^4 = \{\tilde{N}_{op}\}$, $Z_p^4 = \tilde{K}_{op}$, gives the special case

$$S_p^4 = S(\{\tilde{N}_{op}\}, \tilde{K}_{op}, L_p) = E(L_p)$$

Thus, in situations in which it is difficult to determine the distribution of L_m, but information about the distribution of passage times with respect to subproject-graphs is available the representation given here allows the construction of estimators which acording to the amount of work one is willing to do will substantially improve all optimistic estimation possibilities known from the literature. The independence condition for the arc passage times can be weakened(see chapter 2.) to the assumption of independence between $X_{K_i}, i = 1,...,\varrho$, for appropriate partitions $\{K_1,...,K_\varrho\}$ of K as can be seen from the theorems. For applications additional independence between $X_{N_{ip}}$, $N_{ip} \in \mathfrak{N}_p$ is useful and realistic because of (3.2).

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