A Stochastic Flow Problem*

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ABSTRACT

If the demand/supply values at the nodes of a given graph are assumed to be random variables standard flow theory is no longer meaningful. A two stage stochastic programming approach can be used to yield an optimal "quasi"-flow solution which minimizes "quasi"-flow costs and expected costs for compensating nonconformity with the actual realizations of the demand/supply. A special case of this formulation is shown to be the well-known stochastic transportation problem.

An example is included for illustration.

1. INTRODUCTION

If problem data are assumed non-stochastic it is well-known how graph theory can be applied to various formulations concerning communication, transportation or flow problems, but unfortunately this assumption is inadequate for many realistic situations of this kind and also for other problems allowing formulation by graph-theoretical tools, thus, some efforts in combining stochastic ond graphtheoretical aspects have been made, see *e.g* [3] which also contains a chapter on flow estimation problems. Whereas in that chapter stochastic linear model theory is used to get for a given graph estimators for the arc flow values satisfying flow conservation constraints for a set of nodes for which demand/supply values are known, in this paper the demand/supply values at the nodes of the given graph are assumed to be random variables the

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realizations of which need not to fulfill the conditions necessary for the existence of flows. Nevertheless, one wants to determine a "quasi"-flow which minimizes "quasi"-flow costs (the exact definition of "quasi"-flow is given in section 2) and expected costs for compensating non-conformity with the actual realizations of the demand/supply. A two stage stochastic programming approach is used allowing to determine the solution by a sequence of properly chosen out-of-kilter circulation problems in a slightly modified graph if the underlying probability distribution is discrete. Situations using more general distribution functions can be reduced to this case, see *e.g.* [4, 6] for approximations of continuous distributions functions by discrete ones.

A well-known special case of the here described approach is the stochastic transportation problem.

For an introduction to stochastic programming see [5] and for an extensive bibliography on papers concerning various topics of stochastic programming [7] the necessary graphtheoretical tools can be found in [1, 2] some of these known definitions, however, which relate particulary to this paper are given in section 2. Section 3 describes the solution procedure and gives needed statements the proofs of which are postponed to section 4. In section 5 an example of simplest form is explicitly illustrated.

2. FORMULATION OF THE PROBLEM

To start with consider the non-stochastic case. Let

$$G=(N, A, I) \tag{1}$$

be a finite, directed, (weakly) connected graph where N (with $|N|=n\in\mathbf{N}$) describes the nodes, A (with $|A|=m\in\mathbf{N}\cup\{0\}$) the arcs, and $I=(I^1, I^2)$ with $I^j: A\rightarrow N$, $j=1, 2$, the incidence mapping where $I^1(a)$, $I^2(a)$ gives the starting-, end node of $a\in A$. For $i, j\in N$ define $A_{ij}=\{a \mid a\in A, I^1(a)=i, I^2(a)=j\}$, if $|A_{ij}|=1$ one can also use (i, j) to denote the single arc from i to j.

Sometimes cost- and capacity-functions $k, c^1, c^2: A\rightarrow\mathbf{R}$, $0\leqslant c^1\leqslant c^2$, are defined on the arcs of G.

In this paper every $f: A\rightarrow\mathbf{R}$ will be called "quasi"-flow on G, additionally, capacity-feasible "quasi"-flow if

$$c_a^1\leqslant f_a\leqslant c_a^2, \quad a\in A$$

holds. Note, that for

$$u_i^G(f) = \sum_{\{a \mid a \in A,\, I^1(a)=i\}} f_a - \sum_{\{a \mid a \in A,\, I^2(a)=i\}} f_a,\ i \in N \quad (2)$$

it follows

$$\sum_{i \in N} u_i^G(f) = 0 \text{ for all "quasi"-flows } f. \quad (3)$$

If one has a partition (Q, R, S) of N and $v : N \to \mathbf{R}$ with

$$v_i > 0,\ i \in Q,\ v_i = 0,\ i \in R,\ v_i < 0,\ i \in S \quad (4{\cdot}1)$$

$$\sum_{i \in Q} v_i = - \sum_{i \in S} v_i \quad (4{\cdot}2)$$

one is interested in a (capacity-feasible) "quasi"-flow satisfying

$$u_i^G(f) = v_i,\ i \in N \quad (5)$$

called (capacity-feasible) $(v_i,\ i \in N)$-flow.

A $(0,\ldots,0)$-flow, that means $Q=S=\phi$, is called circulation in G.

Well-known (non-stochastic) flow problems *e.g.*

- determine capacity-feasible $(v_1,\ldots,v_n)$-flows which minimize a given cost functional $\sum_{a \in A} k_a f_a$, or,
- look for such flows which among all capacity-feasible $(v_1,\ldots v_n)$-flows for arbitrary v satisfying (4) for a given partition (Q, R, S) with $Q \neq \phi$ maximize $\sum_{i \in Q} v_i$.

For node $i \in N$, v_i can be interpreted as supply/demand according to whether $i \in Q/i \in S$, and it appears realistic to assume that these values are given by random variables $\underset{\sim}{v_i}$ defined on a given probability space (Ω, F, Pr).

In the special case if only for $i \in S$, $\underset{\sim}{v_i}$ are allowed to be (non-positive) random variables one gets the well-known stochastic transportation problem (with stochastic demand). As for given $\omega \in \Omega$, the realization $\tilde{v}_i(\omega)$, $i \in N$, will determine the partition $(Q(\omega), R(\omega), S(\omega))$ of N according to (4.1) and, generally, will violate the condition (4.2)

$$\sum_{i \in Q(\omega)} \tilde{v}_i(\omega) + \sum_{i \in S(\omega)} \tilde{v}_i(\omega) = 0$$

$(\tilde{v}_1(\omega), \ldots, \tilde{v}_n(\omega))$-flows will not exist, but one can look for a "quasi"-flow f (capacity-feasible if needed) and compensate the possible non-conformity

$$u_i^G(f) \neq \tilde{v}_i(\omega)$$

by adequate means at corresponding costs.

In this paper a "two stage stochastic programming" approach is used to handle this problem.

Let there be given linear costs for compensating non-conformity with a "quasi"-flow f defined for fixed $i \in N$, $\omega \in \Omega$, by

$$\varphi_i^f(\tilde{v}_i(\omega)) = \begin{cases} -\gamma_i\,(\tilde{v}_i(\omega) - u_i^G(f)) & < \\ 0 & \text{if } \tilde{v}_i(\omega) = u_i^G(f) \\ \delta_i\,(\tilde{v}_i(\omega) - u_i^G(f)) & > \end{cases} \tag{6}$$

where δ_i, γ_i are given real numbers satisfying

$$\delta_i + \gamma_i > 0, \; i \in N \tag{7}$$

φ_i^f is a random variable on (Ω, F, Pr) for every "quasi"-flow f. Let E denote expectation, then the aim of this paper is to find a solution for the following "stochastic flow problem"

$$\sum_{a \in A} k_a f_a + \sum_{i \in N} E(\varphi_i^f) = \min\,!$$

$$\overset{1}{c}_a \leqslant f_a \leqslant \overset{2}{c}_a,\; a \in A \tag{8}$$

that is, to look for a capacity-feasible "quasi"-flow which minimizes "quasi"-flow costs and expected costs for compensating non-conformity with the actual realizations of demand/supply. For computational purposes we will consider (8) for cases when the support of $\underset{\sim}{v_i}$ is finite which is basic for considerations concerning approximations of more general (known) distribution functions, see *e.g.* the already mentioned articles [4], [6], and also for situations using the empirical distribution of $\underset{\sim}{v_i}$ when the actual distribution function is unknown.

Let

$$v_{i1} < v_{i2} < \ldots\ldots < v_{ir_i},\ v_{ir} \in \mathbf{R},\ i \in N,\ r = 1, \ldots\ldots, r_i \qquad (9{\cdot}1)$$

$$Pr(\underset{\sim}{v_i} = v_{ir}) > 0,\ \sum_{r=1}^{r_i} Pr(\underset{\sim}{v_i} = v_{ir}) = 1 \qquad (9{\cdot}2)$$

$$\text{there exists } i^* \in N \text{ with } v_{i^*r} \neq 0,\ r = 1, \ldots\ldots, r_{i^*} \qquad (9{\cdot}3)$$

where (9·3) means that for every realization $\underset{\sim}{v_i}(\omega)$, $i \in N$, one has $Q(\omega) \cup S(\omega) \neq \emptyset$. Now, because of (9·1), (9·2) the expectation term can be written as a finite sum

$$E(\varphi_i^f) = \sum_{r=1}^{r_i} \varphi_i^f(v_{ir}) Pr(\underset{\sim}{v_i} = v_{ir})$$

and each $\varphi_i^f(v_{ir})$ can be realized as optimal value of the following linear (recourse) program

$$\varphi_i^f(v_{ir}) = \min \{\delta_i\, y_i^+ + \gamma_i\, y_i^- \mid y_i^+ - y_i^- = v_{ir} - u_i^G(f),\ y_i^+, y_i^- \geqslant 0\}$$

allowing to rewrite (8) as a linear program of the form

$$\begin{aligned}
&\sum_{a \in A} k_a f_a + \sum_{i \in N} \sum_{r=1}^{r_i} (\delta_i y_{ir}^+ + \gamma_i y_{ir}^-) Pr(\underset{\sim}{v_i} = v_{ir}) = \min\,! \\
&u_i^G(f) + y_{ir}^+ - y_{ir}^- = v_{ir} \qquad , i \in N,\ r \in 1, \ldots, r_i \\
&c_a^1 \leqslant f_a \leqslant c_a^2 \qquad , a \in A \\
&y_{ir}^+, y_{ir}^- \geqslant 0
\end{aligned} \qquad (10)$$

The dual program of (10) can be written in the form

$$\sum_{i \in N} \sum_{r=1}^{r_i} v_{ir} \pi_{ir} + \sum_{a \in A} c_a^1 \epsilon_a^1 - \sum_{a \in A} c_a^2 \epsilon_a^2 = \max !$$

$$\sum_{r=1}^{r_i} \pi_{ir} - \sum_{r=1}^{r_j} \pi_{jr} + \epsilon_a^1 - \epsilon_a^2 = k_a, \; a \in A \text{ with } \begin{matrix} I^1(a)=i \\ I^2(a)=j \end{matrix} \tag{11}$$

$$-\gamma_i Pr(\underset{\sim}{v_i} = v_{ir}) \leqslant \pi_{ir} \leqslant \delta_i Pr(\underset{\sim}{v_i} = v_{ir}) \qquad \begin{matrix} i \in N \\ r = 1,2,\ldots,r_1 \end{matrix}$$

$$\epsilon_a^1, \; \epsilon_a^2 \geqslant 0.$$

Of course, these linear programs could be solved by versions of the simplex method but depending on the number r_i of realizations of $\underset{\sim}{v_i}$ the dimensions of the problems (at least, when using approximation arguments) could be too large. This is also the main problem for stochastic programming formulations of this kind, see *e.g.* remarks given in [4], [6].

In the next section we will show how problem (10) can be solved by a finite sequence of out-of-kilter circulation problems (see (20)) on graphs which are of about the same size as the original one. To ensure finiteness restriction to rational data is made for the capacities c^1_a, c^2_a and costs k_a, $a \in A$, as well as for the realizations v_{ir}, and their probabilities, $i \in N$, $r=1,\ldots,r_i$.

3. SOLUTION PROCEDURE

Because of the capacity restrictions there exist numbers m_i, M_i such that

$$m_i \leqslant u_i^G(f) \leqslant M_i, \; i \in N$$

holds for all capacity-feasible "quasi"-flows f. Select finite values v_{i0}, $v_{i(r_i+1)}$ in such a way that

$$\left.\begin{matrix} v_{i0} < \min\{v_{i1}, m_i\} \\ v_{i(r_i+1)} > \max\{v_{ir_i}, M_i\} \end{matrix}\right\}, \; i \in N \tag{12}$$

is satisfied. These quantities are introduced for computational convenience, one shall see later what role they will play in the solution procedure.

Now, for each node $i \in N$ select one realization of $\underset{\sim}{v_i}$, say $v_{i\mathbf{x}_i}$ with $\mathbf{x}_i \in \{0, 1,...,r_i\}$ (see also (12)) and define vectors $g, h \in \mathbf{R}^n$ the i-th component of which is given by

$$g_i = v_{i\mathbf{x}_i},\ h_i = v_{i(\mathbf{x}_i+1)} \tag{13}$$

These vectors g, h lead to a partition (N_1, N_2, N_3) of N where

$$\begin{aligned} N_1 &= \{i \mid i \in N,\ g_i > 0\} \\ N_2 &= \{i \mid i \in N,\ h_i < 0\} \\ N_3 &= N \setminus (N_1 \cup N_2) \end{aligned} \tag{14}$$

and allow the construction of the following graph $G_1^{(g, h)}$ which is a slight modified version of the underlying graph G

$$G^{(g,h)} = (N^{(g,h)}, A^{(g,h)}, I^{(g,h)}) \tag{15}$$

where

$N^{(g,h)} = N \cup \{q, s\}$ with $q, s \notin N$ for all g, h of (13)

and

$$A^{(g,h)} = A \cup \bigcup_{l=0}^{3} A_l$$

and for which to simplify notation the incidence mapping $I^{(g,h)}$ is omitted because on A one can use I and the additional arcs of $\bigcup_{l=0}^{3} A_l$ can uniquely be identified by its starting- and end-nodes in the following way:

$$\begin{aligned} A_0 &= \{(s, q)\} \\ A_1 &= \{(q, i) \mid i \in N_1\} \\ A_2 &= \{(i, s) \mid i \in N_2\} \\ A_3 &= \{(q, i), (i, s) \mid i \in N_3\} \end{aligned} \tag{16}$$

Capacity-constraints $c^{(g,h)1}, c^{(g,h)2} : A^{(g,h)} \to \mathbf{R}$ with $0 \leqslant c^{(g,h)1} \leqslant c^{(g,h)2}$ are given by

$$c_a^{(g, h)1} = c_a^1,\quad c_a^{(g, h)2} = c_a^2 \qquad , a \in A$$

$$
\begin{aligned}
&c^{(g,h)1}_{(s,q)}=0, \quad c^{(g,h)2}_{(s,q)}=\sum_{i\in \mathbf{N}} \max\{|v_{i0}|, |v_{i(r_i+1)}|\}, && (s,q)=A_0 \\
&c^{(g,h)1}_{(q,i)}=g_i, \quad c^{(g,h)2}_{(q,i)}=h_i && ,\ (q,i)\in A_1 \qquad (17)\\
&c^{(g,h)1}_{(i,s)}=-h_i, \quad c^{(g,h)2}_{(i,s)}=-g_i && ,\ (i,s)\in A_2 \\
&c^{(g,h)1}_{a}=0, \quad c^{(g,h)2}_{a}=\begin{cases} h_i, & a=(q,i)\\ -g_i, & a=(i,s)\end{cases} && ,\ a\in A_3
\end{aligned}
$$

With

$$\rho_i=-\delta_i+(\delta_i+\gamma_i)Pr(\underset{\sim}{v_i}\leqslant g_i),\ i\in N \tag{18}$$

where (with $g_i=v_{i\mathbf{x}_i}$, see (13))

$$Pr(\underset{\sim}{v_i}\leqslant g_i)=\begin{cases} 0 & \mathbf{x}_i=0\\ \sum_{r=1}^{\mathbf{x}_i} Pr(\underset{\sim}{v_i}=v_{ir}) & \mathbf{x}_i\geqslant 1\end{cases}$$

costs with respect to $G^{(g,h)}$ are yielded by

$$k_a^{(g,h)}=\begin{cases} k_a+\rho_i-\rho_j & \text{if } a\in A \text{ with } I^1(a)=i,\ I^2(a)=j\\ 0 & \text{if } a\in \bigcup_{l=0}^{3} A_l\end{cases} \tag{19}$$

This construction is illustrated by an example of simplest form, see section 5.

For $G^{(g,h)}$ the following minimal cost flow problem (given in circulation form) is of interest which allows application of known out-of-kilter solution procedures, see *e.g.* [1], [2].

$$
\begin{aligned}
&\sum_{a\in A} k_a^{(g,h)} f'_a = \min!\\
&u_i^{G^{(g,h)}}(f')=0, \quad i\in N^{(g,h)} \qquad (20)\\
&c_a^{(g,h)1}\leqslant f'_a\leqslant c_a^{(g,h)2},\ a\in A^{(g,h)}
\end{aligned}
$$

The dual program of (20) is given by

$$\sum_{a\in A} (g, h)\, c_a^{(g,\, h)1} \alpha_a - \sum_{a\in A} (g, h)\, c_a^{(g,\, h)2} \beta_a = \max\,!$$

$$w_i - w_j + \alpha_a - \beta_a = k_a^{(g,\, h)}, \quad a \in A^{(g,h)} \text{ with } \quad \begin{array}{r} a=(i, j) \text{ if } a \notin A \\ \text{or } I^1(a)=i,\ I^2(a)=j \\ \text{if } a \in A \end{array} \qquad \ldots (21)$$

$$\alpha_a, \beta_a \geqslant 0$$

The connection between (8) or (10) and (20) is given by

LEMMA 1. *For every capacity feasible "quasi"-flow f on G according to (8) satisfying*

$$g_i \leqslant u_i^G(f) \leqslant h_i, \quad i \in N \tag{22}$$

there exists a capacity-feasible circulation f' on $G^{(g,h)}$ according to (20) such that

$$\sum_{a\in A} k_a^{(g,\, h)} f'_a = \sum_{a\in A} k_a f_a + \sum_{i\in N} E(\varphi_i^f) - B^{(g,\, h)} \tag{23}$$

where $B^{(g,h)}$ does not depend on f (and f').

Conversly, for every capacity-feasible circulation f' on $G^{(g,h)}$ according to (20) there exists a capacity-feasible "quasi"-flow f on G according to (8) such that (22), (23) holds.

PROOF. See section 4.

REMARK. We use the term "corresponding to" to denote the "quasi"-flow f on G yielded from a circulation f' on $G^{(g,h)}$ by $f=f'/A$ and, *vice versa, a* circulation f' on $G^{(g,h)}$ yielded from a "quasi"-flow f on G in such a way that $f'/A=f$.

Applying the out-of-kilter algorithm to (20) will determine an optimal circulation $\tilde{f}'$ (if there exists one) and the "quasi"-flow $\tilde{f}$ corresponding to $\tilde{f}'$ will be a best one satisfying (22). Moreover, the out-of-kilter method will also produce certain dual quantities w_i, $i \in N^{(g,h)}$ (and α_a, β_a), see (21), which allow to formulate a sufficient condition for the "quasi-flow $\tilde{f}$ corresponding to $\tilde{f}'$ to be a best one at all.

Define for given w_i, $i \in N^{(g,h)}$

$$\alpha_a = \max\ \{0, -(w_{I^1(a)} - w_{I^2(a)} - k_a^{(g,h)})\} \quad , \quad a \in A$$

$$\beta_a = \max\ \{0, w_{I^1(a)} - w_{I^2(a)} - k_a^{(g,h)}\} \tag{24}$$

$$\left.\begin{array}{l} \alpha_{(i,j)} = \max\ \{0, w_j - w_i\} \\ \beta_{(i,j)} = \max\ \{0, w_i - w_j\} \end{array}\right\} , \quad (i,j) \in \bigcup_{\ell=0}^{3} A_\ell$$

and

$$\lambda_i = \begin{cases} \alpha_{(q,i)} & , \ i \in N_1 \\ \beta_{(i,s)} & , \ i \in N_2 \cup N_3 \end{cases} \tag{25}$$

$$\mu_i = \begin{cases} \beta_{(q,i)} & , \ i \in N_1 \cup N_3 \\ \alpha_{(i,s)} & , \ i \in N_2 \end{cases} \tag{26}$$

The following notation is known from out-of-kilter theory:

For a circulation f' according to (20) and arbitrary w_i, $i \in N^{(g,h)}$, call $a \in A^{(g,h)}$ in kilter iff

$$\begin{array}{l} \alpha_a > 0 \Rightarrow f'_a = c_a^{(g,h)1}, \ \beta_a > 0 \Rightarrow f'_a = c_a^{(g,h)2} \\ \alpha_a = \beta_a = 0 \Rightarrow c_a^{(g,h)1} \leqslant f_a \leqslant c_a^{(g,h)2} \end{array} \tag{27}$$

otherwise, call $a \in A^{(g,h)}$ out-of-kilter.

If (20) has feasible solutions then for $\tilde{f}'$ and $\tilde{w}_i$, $i \in N^{(g,h)}$, determined in the last step of the out-of-kilter method when all arcs are in-kilter, and $\tilde{\mu}_i$, $\tilde{\lambda}_i$ (which are given by $\tilde{w}_i$, $i \in N^{(g,h)}$, according to (25), (26)), $i \in N$, one has

THEOREM 1. *If*

$$\left.\begin{array}{l} \tilde{\lambda}_i \leqslant (\delta_i + \gamma_i) Pr\ (\nu_i = g_i) \\ \tilde{\mu}_i \leqslant (\delta_i + \gamma_i) Pr\ (\tilde{\nu}_i = h_i) \end{array}\right\} , \quad i \in N \tag{28}$$

then the "quasi"-flow $\tilde{f}$ corresponding to the optimal circulation $\tilde{f}'$ obtained by the out-of-kilter algorithm is optimal for (8).

PROOF. See section 4.

Now assume that (28) is not satisfied, then, from (7), (24), (25) and (26) we get a partition $(N_O, N_\oplus, N_\ominus)$ of N with

$$N_O=\{i \mid i \in N, \text{(28) is satisfied}\}$$

$$N_\oplus=\{i \mid i \in N, \tilde{\mu}_i > (\delta_i+\gamma_i)\, Pr\,(\tilde{v}_i=h_i)\} \tag{29}$$

$$N_\ominus=\{i \mid i \in N, \tilde{\lambda}_i > (\delta_i+\gamma_i)\, Pr\,(\tilde{v}_i=g_i)\}$$

(For checking that $(N_O, N_\oplus, N_\ominus)$ is a partition of N use also $\tilde{w}_a=\tilde{w}_s$ shown in the proof of Theorem 1, see (52), section 4). To construct a new circulation problem and a new graph $G^{(\bar{q}, \bar{h})}$ define

$$\bar{x}_i=\begin{cases} x_i \quad , i \in N_O \\ x_i+1, \ i \in N_\oplus \\ x_i-1, \ i \in N_\ominus \end{cases} \tag{30}$$

where (12) guarantees that $\bar{x}_i \in \{0, 1, \ldots, r_i\}$ by the following arguments :

$$x_i=0 \text{ and } i \in N_\ominus \cap N_1 \underset{(29)}{\overset{(25)}{\Rightarrow}} \tilde{\lambda}_i=\tilde{\alpha}_{(q,i)} > 0 \underset{(27)}{\overset{(14),\,(17)}{\Rightarrow}} \tilde{f}'_{(q,i)}$$

$=c^{(\bar{q},\bar{h})1}_{(q,i)}=v_{i0}$, but $\tilde{f}'$ being a circulation gives

$$\tilde{f}'_{(q,i)}=\sum_{\{a \mid a\in A,\, \Gamma^1(a)=i\}} \tilde{f}_a' - \sum_{\{a \mid a\in A,\, \Gamma^2(a)=i\}} \tilde{f}_a'$$

$$=u_i^q\,(\tilde{f})=v_{i0}$$

for the capacity-feasible "quasi"-flow $\tilde{f}$ corresponding to $\tilde{f}'$, a contradiction to (12). The cases $x_i=0$, $i \in N_\ominus \cap N_2$ and $x_i=0$, $i \in N_\ominus \cap N_3$

can be treated in the same way, thus

$$x_i=0 \Rightarrow i \notin N_\ominus.$$

The proof of

$$x_i=r_i \Rightarrow i \notin N_\oplus$$

can be yielded by similar arguments and is omitted.

Now, use $\bar{g}$, $\bar{h}$, $\bar{\rho}$ to denote the quantities given by $\bar{x}_i$ according to (13), (18). With these quantities create the new graph $G^{(\bar{g}, \bar{h})}$ and the new minimum circulation problem, see also (20)

$$\sum_{a\in A} k_a^{(\bar{g}, \bar{h})} f_a' = \min !$$

$$u_i^{G^{(\bar{g}, \bar{h})}}(f')=0, \; i \in N^{(\bar{g}, \bar{h})} \tag{31}$$

$$c_a^{(\bar{g}, \bar{h})1} \leqslant f_a' \leqslant c_a^{(\bar{g}, \bar{h})1}, \; a \in A^{(\bar{g}, \bar{h})}$$

Taking the optimal circulation $\tilde{f}'$ yielded by application of the out-of-kilter algorithm to (20) and defining $\bar{f}'$ by

$$\bar{f}_a' = \begin{cases} \tilde{f}_a' & , \; a \in A^{(g, h)} \cap A^{(\bar{g}, \bar{h})} \\ 0 & , \; \text{otherwise} \end{cases} \tag{32}$$

gives

LEMMA 2. *$\bar{f}'$ is capacity-feasible circulation in $G^{(\bar{g}, \bar{h})}$ according to (31) with $\tilde{f}'/A=\bar{f}'/A$.*

PROOF. See section 4.

Moreover, defining $\bar{w}_i$ by

$$\bar{w}_i = \begin{cases} \tilde{w}_i & , \; i \in N_O \cup \{q, s\} \\ \tilde{w}^i + (\delta_i+\gamma_i) Pr(\underset{\sim}{v_i}=h_i) & , \; i \in N_\oplus \\ \tilde{w}_i - (\delta_i+\gamma_i) Pr(\underset{\sim}{v_i}=g_i) & , \; i \in N_\ominus \end{cases} \tag{33}$$

gives

LEMMA 3. *For* $\bar{f}'$ *and* $\bar{w}_i$, $i \in N^{(\bar{g}, \bar{h})}$ $(=N^{(g, h)})$, *used as starting quantities for (31), it follows*

(*i*) $a \in A \cup A_O \Rightarrow$ *a is in-kilter arc*

(*ii*) $i \in N_O \Rightarrow (q, i), (i, s) \in A^{(\bar{g}, \bar{h})}$ *are in-kilter arcs*

(*iii*) $i \in N_\oplus \cup N_\ominus \Rightarrow$ *not all* $(q, i), (i, s) \in A^{(\bar{g}, \bar{h})}$ *are in-kilter arcs.*

PROOF. See section 4.

In view of Lemma 2 the minimum cost circulation problem (31) has a feasible and therefore an optimal solution.

Moreover Lemma 3 assures the additional "kilter" steps have to be performed when applying the out-of-kilter algorithm to (31) using $\bar{f}'$ and $\bar{w}_i$, $i \in N^{(\bar{g}, \bar{h})}$ given by (32), (33) as starting quantities.

The following lemma is needed.

LEMMA 4. *Let for an underlying graph* $G^{(g, h)}$ f' *be a capacity-feasible circulation and* w_i, $i \in N^{(g, h)}$, *be given in such a way that all arcs of* $A \cup A_0$ *are in-kilter.*

If there exists an (elementary) cycle C (containing at least one out-of-kilter arc) for which a circulation change of value $\triangle > 0$ *according to kilter-rules is possible, then*

$$\sum_{a \in A} k_a^{(g, h)}(f_O')_a < 0$$

where $f_O' : A^{(g, h)} \rightarrow \mathbf{R}$ *with*
$$(f_O')_a = \begin{cases} \triangle, & a \in A^+(C) \\ -\triangle, & a \in A^-(C) \\ 0, & a \notin A(C) \end{cases}$$
describes the circulation change, $(A^+(C), A^-(C))$ *the partition of the cycle arc-set* $A(C)$ *indicating coincidence of arc and circulation change directions.*

PROOF. See section 4.

Denoted by $\hat{f}'$ the optimal capacity-feasible circulation of (31) determined by application of the out-of-kilter algorithm with starting quantities $\bar{f}', \bar{w}_i, i \in N^{(\bar{g}, \bar{h})}$.

Using Lemma 4 one can show

THEOREM 2. *If $\hat{f}' \neq \bar{f}'$, then*

$$\sum_{a \in A} k_a^{(\bar{g}, \bar{h})} \hat{f}_a' < \sum_{a \in A} k_a^{(\bar{g}, \bar{h})} \bar{f}_a' \tag{34}$$

and $\tilde{f}' = \bar{f}'/A$ is not optimal in (8).

PROOF. See section 4.

Thus, if $\hat{f}' \neq \bar{f}'$ the construction of the new indices $\bar{x}_i, i \in N$, according to (30) has led to a better "quasi"-flow solution to (8), and *restarting* the whole procedure with $\bar{x}_i, i \in N$, a better solution of (8) (which is $\hat{f}'/A$, as we already know) will be found.

On the other hand if $\hat{f}' = \bar{f}'$ (this case corresponds to degeneracy in linear programming theory) a "modified version of the out-of-kilter algorithm" has to be applied to make sure whether an improvement is still possible, which is indicated by creating a new graph.

Modified version of the out-of-kilter algorithm :

Starting on $G^{(\bar{g}, \bar{h})}$ with the known quantities $\bar{f}_i', \bar{w}_i, i \in N^{(\bar{g}, \bar{h})}$, which are given at the beginnig of the modified version by (32), (33), and in further steps (if no new graph is created) by the updating according to (47), see below, and using a starting out-of-kilter arc (q, j) or (j, s), $j \in N_{\oplus} \cup N_{\ominus}$, two cases have to be distinguished :

If $j \in N_{\oplus}$ define

$$\bar{X}=\{j\} \cup \{i \in N^{(\bar{g}, \bar{h})} \mid \text{there exists a path } P_{ji} \text{ from } j \text{ to } i \text{ which allows arc flow alteration according to kilter-rules}\} \quad (35)$$

$$\bar{Y}=N^{(\bar{g}, \bar{h})} \setminus \bar{X}$$

$$\Theta_0=\min \{(\delta_l+\gamma_l)Pr(\underset{\sim}{v_l}=\bar{g}_l)-\bar{\lambda}_l \mid l \in N_0 \cap \bar{X}, \bar{\mu}_l=0\} \quad (36)$$

If $j \in N_{\ominus}$ define

$$\bar{Y}=\{j\} \cup \{i \in N^{(\bar{g}, \bar{h})} \mid \text{there exists a path } P_{ij} \text{ from } i \text{ to } j \text{ which allows arc-flow alteration according to kilter-rules}\} \quad (37)$$

$$\bar{X}=N^{(\bar{g}, \bar{h})} \setminus \bar{Y}$$

$$\Theta_0=\min \{(\delta_l+\gamma_l)Pr(\underset{\sim}{v_l}=\bar{h}_l)-\bar{\mu}_l \mid l \in N_0 \cap \bar{Y}, \bar{\lambda}_l=0\} \quad (38)$$

Denote for $M_1, M_2 \subset N^{(\bar{g}, \bar{h})}$ by

$$(M_1, M_2)=\{a \mid a \in A^{(\bar{g}, \bar{h})}, a=(i, j), i \in M_1, j \in M_2\}$$

the set of arcs which have their starting nodes in M_1, their end nodes in M_2 and define, similar as in the dual phase of the standard out-of-kilter algorithm (see [1])]

$$S_1=\{a \in A^{(\bar{g}, \bar{h})} \mid \bar{\alpha}_a>0, a \in (\bar{X}, \bar{Y})\} \quad (39)$$

$$S_2=\{a \in A^{(\bar{g}, \bar{h})} \mid \bar{\beta}_a>0, a \in (\bar{Y}, \bar{X})\}$$

$$\Theta_{(\bar{X}, \bar{Y})}=\min\{\min \{\bar{\alpha}_a \mid a \in S_1\}, \min \{\bar{\beta}_a \mid a \in S_2\}\} \quad (40)$$

and

$$\Theta^*=\min \{\Theta_0, \Theta_{(\bar{X}, \bar{Y})}\} \quad (41)$$

and change the dual quantities according to

$$w_i^* = \begin{cases} \bar{w}_i + \Theta^* & , \ i \in \bar{X} \\ \bar{w}_i & , \ i \in \bar{Y}. \end{cases} \tag{42}$$

If

$$0 < \Theta^* = \Theta_{(\bar{X}, \bar{Y})} \leqslant \Theta_0 \tag{43}$$

the dual change according to (42) coincides with the dual phase of the standard out-of-kilter algorithm. Additionally, if

$$\alpha^*_{(q, i_0)} = \beta^*_{(q, i_0)} = 0 \text{ or/and } \alpha^*_{(i_0, s)} = \beta^*_{(i_0, s)} = 0 \tag{44}$$

for some arc (q, i_0) or/and (i_0, s) with $i_0 \in N_{\oplus} \cup N_{\ominus}$

$$i_0 \text{ is added to } N_0. \tag{45}$$

If

$$0 \leqslant \Theta^* = \Theta_0 < \Theta_{(\bar{X}, \bar{Y})} \tag{46}$$

after a possible dual change according to (42) (if $\Theta^* > 0$) a new graph $G^{(\bar{g}, \bar{h})}$ can be defined and the algorithm stops.

Whenever no new graph is created the following *updating*

$$w_i := w_i^* \tag{47}$$

$N_0 := N_0 \cup \{i_0\}$ if (44) holds,

the corresponding $N_{\oplus}$ or $N_{\ominus}$ set is reduced

is performed, and a new dual phase of the modified version is started working on the same starting out-of-kilter arc until it becomes in-kilter. If no out-of-kilter arc is left, the algorithm stops.

LEMMA 5. *During all steps of the "modified out-of-kilter algorithm"*

$$\bar{w}_q = \bar{w}_s \tag{48}$$

(i.e. $\bar{\alpha}_{(q, s)} = \bar{\beta}_{(s, q)} = 0$*) is satisfied.*

PROOF. See section 4.

THEOREM 3. *If $\hat{f}'=\bar{f}'$ application of the "modified version of the out-of-kilter algorithm" leads to*

either (a) *new quantities $\hat{w}_i$, $i \in N^{(\bar{g}, \bar{h})}$, such that for $\hat{f}'$, $\hat{w}_i$, $i \in N^{(\bar{g}, \bar{h})}$, all arcs $a \in A^{(\bar{g}, \bar{h})}$ are in-kilter and for the corresponding $\hat{\lambda}_i$, $\hat{\mu}_i$, $i \in N$, (28) is satisfied*

or (b) *new quantities $\bar{\bar{\mathbf{x}}}_i$, $i \in N$, determining a new graph $G^{(\bar{\bar{g}}, \bar{\bar{h}})}$ and a new minimum cost circulation problem according to $G^{(\bar{\bar{g}}, \bar{\bar{h}})}$ such that there exists a capacity-feasible circulation corresponding to $\tilde{f}=\hat{f}'|A$ which is not optimal.*

PROOF. See section 4.

Thus, if $\bar{f}'=\hat{f}'$ and the "modified version of the out-of-kilter algorithm" has led to a new graph $G^{(\bar{\bar{g}}, \bar{\bar{h}})}$ (Theorem 3 (*b*)) *restarting* the whole procedure with $\bar{\bar{\mathbf{x}}}_i$, $i \in N$, a better solution of (8) will be found.

In any case the procedure is *restarted* an improvement is possible. Therefore no $(. , \mathbf{x}_i, ..)$-vector is used twice. Because of (9.1), (12) an optimal solution of (8) is determined by solving a finite number of minimum cost circulation problems of the form (20) where application of a "modified version of the out-of-kilter algorithm" may be necessary.

4. PROOFS OF THEOREMS

At the beginning of this section proofs are given in a rather explicit form whereas a more compressive form is used later on. Some knowledge how to handle out-of-kilter algorithms could be helpful, see *e.g.* [1], [2].

PROOF OF LEMMA 1 :

From the construction of $G^{(g, h)}$ it easily follows that every capacity-feasible "quasi"-flow f according to (8) which satisfies (22) can be extended to a capacity-feasible circulation f' according to (20). Set

$$f_a'=f_a \qquad , \quad a \in A \qquad (49\cdot1)$$

$$f'_{(s,q)} = \sum_{i \in N_1} u_i^G(f) + \sum_{i \in N_3} \max\{0, u_i^G(f)\}, \quad (s,q) \in A_0 \qquad \ldots(49{\cdot}2)$$

$$f'_{(q,i)} = u_i^G(f) \qquad , \quad (q,i) \in A_1 \qquad (49{\cdot}3)$$

$$f'_{(i,s)} = -u_i^G(f) \qquad , \quad (i,s) \in A_2 \qquad (49{\cdot}4)$$

$$f_a' = \begin{cases} \max\{0, u_i^G(f)\}, \ a=(q,i) \\ \max\{0, -u_i^G(f)\}, \ a=(i,s) \end{cases} , \ a \in A_3 \qquad (49{\cdot}5)$$

Then it is obvious that f' is capacity-feasible according to (20), see (17), additionally

$$u_i^{G^{(q,h)}}(f') = \begin{cases} u_i^G(f) - f'_{(q,i)} = 0 & \text{if } i \in N_1 \text{ (see (49.3))} \\ f'_{(i,s)} + u_i^G(f) = 0 & \text{if } i \in N_2 \text{ (see (49.4))} \\ f'_{(i,s)} + u_i^G(f) - f'_{(q,i)} = 0 & \text{if } i \in N_3 \text{ (see (49.5))} \\ \sum_{(i,j) \in A_1 \cup A_3} f'_{(i,j)} - f'_{(s,i)} = \sum_{(i,j) \in A_1} f'_{(i,j)} + \sum_{(i,j) \in A_3} f'_{(i,j)} - f'_{(s,i)} = 0 & \text{if } i = q \text{ (see (49.2), (49.3), (49.5))} \end{cases}$$

For the remaining case $i = s$ note that

$$\sum_{(j,s) \in A_2} f'_{(j,s)} = \sum_{j \in N_2} (-u_j^G(f)),$$

$$\sum_{(j,s) \in A_3} f'_{(j,s)} = \sum_{j \in N_3} (-u_j^G(f))$$

$$\text{with } u_j^G(f) < 0$$

and (because of (3))

$$\sum_{j\in N} u_j^G(f) = \sum_{j\in N_1} u_j^G(f) + \sum_{j\in N_2} u_j^G(f) + \underset{\text{with } u_j^G(f)>0}{\sum_{j\in N_3} u_j^G(f)} + \underset{\text{with } u_j^G(f)<0}{\sum_{j\in N_3} u_j^G(f)} = 0$$

thus

$$u_s^{G^{(g,h)}}(f') = f'_{(s,q)} - \sum_{(j,s)\in A_2} f'_{(j,s)} - \sum_{(j,s)\in A_3} f'_{(j,s)} = 0$$

showing that f' is a capacity-feasible circulation according to (20). On the other side if f' is a capacity-feasible circulation according to (20) $f=f'/A$ is a "quasi"-flow by definition and capacity-feasible because the capacity-constraints of G and $G^{(g,h)}$ coincide on A, see (17). For $i\in N$

$$u_i^G(f) = \underbrace{u_i^{G^{(g,h)}}(f')}_{=0} \begin{cases} +f'_{(q,i)} & , i\in N_1 \\ -f'_{(i,s)} & , i\in N_2 \\ +f'_{(q,i)} - f'_{(i,s)} & , i\in N_3 \end{cases}$$

but f' being capacity-feasible according to (20) gives, see (17)

$$\begin{array}{ll} g_i \leqslant f'_{(q,i)} \leqslant h_i & , i\in N_1 \\ -h_i \leqslant f'_{(i,s)} \leqslant -g_i & , i\in N_2 \\ 0-(-g_i) \leqslant f'_{(q,i)} - f'_{(i,s)} \leqslant h_i - 0 & , i\in N_3 \end{array}$$

Now, take a "quasi"-flow f according to (8) satisfying (22) then

$$\varphi_i^f(v_{ir}) = \begin{cases} \gamma_i\,(u_i^G(f) - v_{ir}) & , 1\leqslant r\leqslant x_i \\ \delta_i\,(v_{ir} - u_i^G(f)) & , x_i+1\leqslant r\leqslant r_i \end{cases}$$

and $E(\varphi_i^f) = \sum_{r=1}^{r_i} \varphi_i^f(v_{ir}) Pr(\underset{\sim}{v_i} = v_{ir})$

$$=\sum_{r=1}^{x_i} \gamma_i (u_i^G(f) - v_{ir}) \; Pr(\underset{\sim}{v_i} = v_{ir})$$

$$+ \sum_{r=x_i+1}^{r_i} \delta_i (v_{ir} - u_i^G(f)) Pr(\underset{\sim}{v_i} = v_{ir}) \tag{50}$$

$$= -\gamma_i \sum_{r=1}^{x_i} v_{ir} \, Pr\,(\underset{\sim}{v_i} = v_{ir}) + \delta_i \sum_{r=x_i+1}^{r_i} v_{ir} \, Pr\,(\underset{\sim}{v_i} = v_{ir})$$

$$+ u_i^G(f)(\sum_{r=1}^{x_i} Pr\,(\underset{\sim}{v_i} = v_{ir})(\gamma_i + \delta_i) - \delta_i)$$

But the first two terms are independent of f (and f'), together denoted by $B_i^{(g, h)}$, the last one is equal to $\rho_i \cdot u_i^G(f)$ with ρ_i given by (18). Thus,

$$\sum_{i \in N} E(\varphi_i^f) = \underbrace{\sum_{i \in N} B_i^{(g, h)}} + \sum_{i \in N} \rho_i \, u_i^G(f) \tag{51}$$

$$= B^{(g, h)} + \sum_{a \in A} (\rho_{I^1(a)} - \rho_{I^2(a)}) f_a$$

and

$$\sum_{a \in A} k_a f_a + \sum_{i \in N} E(\varphi_i^f) = \sum_{a \in A} (k_a + \rho_{I^1(a)} - \rho_{I^2(a)}) \, f'_a + B^{(g, h)}$$

Conversely, given a capacity-feasible circulation f' according to (20) take the "quasi"-flow $f = f'/A$ corresponding to f' which satisfies (22) as shown in the beginning of Lemma 1 from which (23) is yielded according to the above arguments. □

Proof of Theorem 1 :

First, observe that

$$\tilde{w}_q = \tilde{w}_s. \tag{52}$$

$\tilde{w}_s - \tilde{w}_q > 0$ would lead to $\tilde{\beta}_{(s, q)} > 0$ and $\tilde{f}'_{(s, q)} = c_{(s, q)}^{(g, h)2}$, see (27) for $a=(s, q)$, but this is impossible by construction because f' being a circulation yields, see (17)

$$\tilde{f}'_{(s,q)} \leqslant \sum_{(i,s)\in A_2 \cup A_3} c^{(g,h)2}_{(i,s)} = \sum_{(i,s)\in A_2 \cup A_3} (-g_i)$$

$$\leqslant \sum_{i\in N} \max.\{|g_i|, |h_i|\} < c^{(g,h)2}_{(s,q)}.$$

On the other hand $\tilde{w}_s - \tilde{w}_q < 0$ would lead to $\tilde{\alpha}_{(s,q)} > 0$ and

$\tilde{f}'_{(s,q)} = c^{(g,h)1}_{(s,q)} = 0$, see (27) for $a=(s, q)$, from which follows

$\tilde{f}'_{(i,s)}=0, \tilde{f}'_{(q,i)}=0$ for all $(i, s), (q, i) \in \bigcup_{l=1}^{3} A_l$, as $\tilde{f}'$ is a circulation. Therefore $N=N_3$, that means $A_1=A_2=\phi$ but, see (27) and (9·3)

$$\left.\begin{array}{l} \tilde{f}'_{(i^*,s)}=0 \text{ gives } \tilde{w}_i{}^* - \tilde{w}_s \leqslant 0 \\ \tilde{f}'_{(q,i^*)}=0 \text{ gives } \tilde{w}_q - \tilde{w}_i{}^* \leqslant 0 \end{array}\right\} \text{thus, } \tilde{w}_i{}^* \leqslant \tilde{w}_s < \tilde{w}_q \leqslant \tilde{w}_i{}^*,$$

a contradiction.

Now, define for $i\in N$

$$\pi_{ir} = \begin{cases} -\gamma_i\, Pr\,(\tilde{v}_i = v_{ir}) & , r \leqslant \mathbf{x}_i - 1 \\ -\gamma_i\, Pr\,(\tilde{v}_i = v_{ir}) + \tilde{\lambda}_i & , r = \mathbf{x}_i \\ \delta_i\, Pr\,(\tilde{v}_i = v_{ir}) - \tilde{\mu}_i & , r = \mathbf{x}_i + 1 \\ \delta_i\, Pr\,(\tilde{v}_i = v_{ir}) & , r \geqq \mathbf{x}_i + 2 \end{cases}$$

then,

$$-\gamma_i\, Pr\,(\tilde{v}_i = v_{ir}) \leqslant \pi_{ir} \leqslant \delta_i\, \text{Pr}\,(\tilde{v}_i = v_{ir}) \tag{54}$$

is valid for $r \notin \{\mathbf{x}_i, \mathbf{x}_i+1\}$ by definition, for $r \in \{\mathbf{x}_i, \mathbf{x}_i+1\}$ by (28) and one can easily verify

$$\sum_{r=1}^{r_i} \pi_{ir} = -\rho_i + \tilde{\lambda}_i - \tilde{\mu}_i \quad , i \in N$$

and, using (52)

$$\tilde{\lambda}_i - \tilde{\mu}_i = \tilde{w}_i - \tilde{w}_q$$

thus,

$$\sum_{r=1}^{r_i} \pi_{ir} = (\tilde{w}_i - \rho_i) - \tilde{w}_q .$$

Defining

$$\epsilon_a^1 = \tilde{\alpha}_a,\ \epsilon_a^2 = \tilde{\beta}_a \ ,\ a \in A \tag{55}$$

one gets for $a \in A$ with $I^1(a)=i$, $I^2(a)=j$, see (19)

$$\sum_{r=1}^{r_i} \pi_{ir} - \sum_{r=1}^{r_j} \pi_{jr} + \epsilon_a^1 - \epsilon_a^2 = \tilde{w}_i - \tilde{w}_j - \rho_i + \rho_j + \tilde{\alpha}_a - \tilde{\beta}_a \tag{56}$$

$$= k_a^{(a,h)} - \rho_i + \rho_j = k_a$$

(54), (56) show that π_{ir}, ϵ_a^1, ϵ_a^2 as given by (53), (55) define a feasible solution of the dual program (11).

On the other side we can define a feasible solution to the primal program (10) by

$$\tilde{f} = \tilde{f}/A$$

$$\overset{+}{y}_{ir} = \begin{cases} 0 \\ v_{ir} - u_i^G(\tilde{f}) \end{cases}, \quad \bar{y}_{ir} = \begin{cases} u_i^G(\tilde{f}) - v_{ir} & , r \leqslant x_i \\ 0 & , r \geqslant x_i + 1 \end{cases} \tag{57}$$

Now showing that for these primal and dual feasible solutions (53), (55), (57) the values of the objective functions of (10) and (11) are equal is sufficient for $\tilde{f}$ to be an optimal 'quasi'-flow solution to (8).

Using considerations as in the proof of Lemma 1, see (50), (51) one gets for the objective function of (10)

$$\sum_{a\in A} k_a \tilde{f}_a + \sum_{i\in N} \Big(\sum_{r=1}^{r_i} \delta_i \overset{+}{y}_{ir} + \gamma_i \overset{-}{y}_{ir}) \Pr (\underset{\sim}{v_i} = v_{ir}) \Big) \tag{58}$$

$$= \sum_{a\in A} (k_a + \rho_{I^1(a)} - \rho_{I^2(a)}) \tilde{f}_a + B^{(g,h)}$$

$$= \sum_{a\in A} k_a^{(g,h)} \tilde{f}_a' + B^{(g,h)}$$

and using duality (notice optimality) for (20), (21)

$$\sum_{a\in A} k_a^{(g,h)} \tilde{f}_a' = \sum_{a\in A^{(g,h)}} c_a^{(g,h)1} \tilde{\alpha}_a - \sum_{a\in A^{(g,h)}} c_a^{(g,h)2} \tilde{\beta}_a . \tag{59}$$

On the other side, for the objective function of (11) one gets from (53), (55)

$$\sum_{i\in N} \sum_{r=1}^{r_i} v_{ir} \pi_{ir} + \sum_{a\in A} c_a^1 \epsilon_a^1 - \sum_{a\in A} c_a^2 \epsilon_a^2$$

$$- \sum_{i\in N} \Big(\sum_{r=1}^{x_i} - \gamma_i v_{ir} \Pr (\underset{\sim}{v_i} = v_{ir}) + \sum_{r=x_i+1}^{r_i} \delta_i v_{ir} \Pr (\underset{\sim}{v_i} = v_{ir}) \Big)$$

$$+ \sum_{i\in N} (\lambda_i v_{ix_i} - \tilde{\mu}_i v_{i(x_i+1)}) + \sum_{a\in A} c_a^1 \tilde{\alpha}_a - \sum_{a\in A} c_a^2 \tilde{\beta}_a \tag{60}$$

Note, that the first sum is equal to $B^{(g,h)}$ which follows from (50), (51), thus, comparing the remaining terms of (58) (expressed with the help of (59)) and (60) it is enough to show equality for

$$\sum_{i\in N} (\tilde{\lambda}_i v_{ix_i} - \tilde{\mu}_i v_{i(x_i+1)})$$

and

$$\sum_{a\in \bigcup_{l=1}^{3} A_l} (c_a^{(g,h)1} \tilde{\alpha}_a - c_a^{(g,h)2} \tilde{\beta}_a)$$

$(s, q) = A_0$ need not to be considered because from (52) $\tilde{\alpha}_{(s,\,q)} = \tilde{\beta}_{(s,\,q)} = 0$. Now, as A_t is defined using N_t, $t=1, 2, 3$, see (16), for

$$i \in N_1: \quad \tilde{\lambda}_i \, v_{ix_i} - \tilde{\mu}_i \, v_{i(x_i+1)} = \tilde{\alpha}_{(q,\,i)} g_i - \tilde{\beta}_{(q,\,i)} \, h_i$$

$$= \tilde{\alpha}_{(q,\,i)} \; c^{(g,\,h)1}_{(q,\,i)} - \tilde{\beta}_{(q,\,i)} \; c^{(g,\,h)2}_{(q,\,i)}$$

$$i \in N_2: \quad \tilde{\lambda}_i \, v_{ix_i} - \tilde{\mu} v_{i(x_i+1)} = \tilde{\beta}_{(i,\,s)} \; g_i - \tilde{\alpha}_{(i,\,s)} \, h_i$$

$$= \tilde{\beta}_{(i,\,s)} \; (-c^{(g,\,h)2}_{(i,\,s)}) - \tilde{\alpha}_{(i,\,s)} \; (-c^{(g,\,h)1}_{(i,\,s)})$$

$$i \in N_3: \quad \tilde{\lambda}_i \, v_{ix_i} - \tilde{\mu}_i v_{i(x_i+1)} = \tilde{\beta}_{(i,\,s)} \, g_i - \tilde{\beta}_{(q,\,i)} \; h_i$$

$$= \tilde{\beta}_{(i,\,s)} \; (-c^{(g,\,h)2}_{(i,\,s)}) \; \tilde{\beta}_{(q,\,i)} \; c^{(g,\,h)2}_{(q,\,i)} + 0 + 0$$

because $\quad c^{(g,\,h)1}_{(i,\,s)} = c^{(g,\,h)1}_{(q,\,i)} = 0.$ □

PROOF OF LEMMA 2:

For $a \in A \cup A_0$ as well as (q, i) and/or (i, s) with $i \in N_0$ the capacity bounds and the arc flows are unchanged by (30), (32).

Let $(\overline{N_1}, \overline{N_2}, \overline{N_3})$ be the partition of N with respect to $\bar{g}$, $\bar{h}$ according to (14) then for $i \in N_{\oplus} \cup N_{\ominus}$ the following cases have to be distinguished (Note that the remaining cases are impossible.):

$$i \in N_1 \cap \overline{N_1} \cap N_{\oplus}, \quad i \in N_1 \cap \overline{N_1} \cap N_{\ominus} \tag{61·1}$$

$$i \in N_1 \cap \overline{N_3} \cap N_{\ominus} \tag{61·2}$$

$$i \in N_2 \cap \overline{N_2} \cap N_{\oplus}, \quad i \in N_2 \cap \overline{N_2} \cap N_{\ominus} \tag{61·3}$$

$$i \in N_2 \cap \overline{N_3} \cap N_{\oplus} \tag{61·4}$$

$$i \in N_3 \cap \overline{N_1} \cap N_{\oplus} \tag{61·5}$$

$$i \in N_3 \cap \overline{N_2} \cap N_{\ominus} \tag{61·6}$$

$$i \in N_3 \cap \overline{N_3} \cap N_{\oplus}, \quad i \in N_3 \cap \overline{N_3} \cap N_{\ominus}. \tag{61·7}$$

Here, only the first part of (61·7) as one of the more complicated cases is explicitly proved. We have

$$(i \in N_3 \Rightarrow h_i \geqslant 0,\ i \in N_{\oplus} \Rightarrow \overline{g_i} = h_i,\ i \in \overline{N_3} \Rightarrow \overline{g_i} \leqslant 0) \Rightarrow h_i = 0 \tag{62}$$

$$i \in N_{\oplus} \Rightarrow \tilde{\mu}_i > 0 \underset{\substack{(17),\,(26),\,(27)\\ i \in N_3}}{\Longrightarrow} \tilde{f}'_{(q,i)} = c^{(g,\,h)2}_{(q,\,i)} = h_i = 0$$

additionally,

$\tilde{\mu}_i > 0$ and $\tilde{w}_q = \tilde{w}_s$, see (52), imply

$$\tilde{\mu}_i = \tilde{\beta}_{(q,\,i)} = \tilde{w}_q - \tilde{w}_i = \tilde{w}_s - \tilde{w}_i = \tilde{\alpha}_{(i,\,s)} > 0 \underset{\substack{(17),\,(26),\,27)\\ i \in N_3}}{\Longrightarrow} \tilde{f}'_{(i,\,s)} = c^{(g,\,h)1}_{(i,\,s)} = 0$$

and, together with (32)

$$\bar{f}'_{(i,\,s)} = \bar{f}'_{(q,\,i)} = 0.$$

For the capacity-constraints one gets

$$i \in \overline{N_3} \underset{(17)}{\Longrightarrow} c^{(\bar{g},\,\bar{h})1}_{(i,\,s)} = c^{(\bar{g},\,\bar{h})1}_{(q,\,i)} = 0$$

$$i \in \overline{N_3} \cap N_{\oplus} \underset{(17),\,(30),\,(62)}{\Longrightarrow} c^{(\bar{g},\,\bar{h})2}_{(i,\,s)} = -\overline{g_i} = -v_{i\bar{x}_i} = -v_i(\bar{x}_i + 1) = -h_i = 0$$

$$c^{(\bar{g},\,\bar{h})2}_{(q,\,i)} = \overline{h_i} = v_i(\bar{x}_i + 1) > 0$$

showing that $\overline{f'}_{(i,\,s)}, \overline{f'}_{(q,\,i)}$ determined in (63) are capacity-feasible arc-flows. To check the circulation condition for $i \in N_3 \cap \overline{N_3} \cap N_{\oplus}$ note that $A \subset A^{(g,h)} \cap A^{(\bar{g},\bar{h})}$ and use (32)

$$u_i^{G^{(\bar{g},\bar{h})}}(\overline{f'}) = u_i^{G}(\overline{f'}/A) + \overline{f'}_{(i,s)} - \overline{f'}_{(q,i)}$$

$$= u_i^{G}(\tilde{f}'/A) + \tilde{f}'_{(i,s)} - \tilde{f}'_{(q,i)} = u_i^{G^{(g,h)}}(\tilde{f}') = 0$$

A discussion of the remaining cases of (61) is similar and omitted. □

PROOF OF LEMMA 3 :

From (18) and (33) follows

$$\overline{w_i} - \overline{\rho_i} = \tilde{w}_i - \rho_i \quad , \quad i \in N$$

and with (19) (and (33))

$$\overline{w_i} - \overline{w_j} - k_a^{(\bar{g},\bar{h})} = \tilde{w}_i - \tilde{w}_j - k_a^{(g,h)} \quad , \quad a \in A \text{ with } \begin{matrix} \Gamma^1(a) = i \\ \Gamma^2(a) = j \end{matrix}$$

$$\overline{w_s} - \overline{w_q} = \tilde{w}_s - \tilde{w}_q \quad , \quad (s, q) = A_0$$

thus, part (*i*) of Lemma 3 is true because $\bar{f}'_{/A \cup A_0} = \tilde{f}'_{/A \cup A_0}$, see (32),

and for $\tilde{f}'$, $\tilde{w}_i$, $i \in N^{(g,h)}$, all arcs have been in-kilter.

Now,

$$i \in N_0 \Rightarrow ((q, i), (i, s) \in A^{(g,h)} \Leftrightarrow (q, i), (i, s) \in A^{(\bar{g},\bar{h})}) \quad (64)$$

additionally, capacity-bounds are not altered for $(q, i), (i, s) \in$

$A^{(g,h)} \cap A^{(\bar{g},\bar{h})}$ and

$\bar{w}_i = \tilde{w}_i$ for $i \in N_0 \cup \{q, s\}$, see (33)

$\bar{w}_q = w_s$ which follows from (52) yields with (24), (32), (64) for $(q, i), (i, s) \in A^{(\bar{g}, \bar{h})}$

$$\bar{\alpha}_{(q,i)} = \tilde{\alpha}_{(q,i)}, \quad \bar{\beta}_{(q,i)} = \tilde{\beta}_{(q,i)} \qquad \bar{f}'_{(q,i)} = \tilde{f}'_{(q,i)}$$

and

$$\bar{\alpha}_{(i,s)} = \tilde{\alpha}_{(i,s)}, \quad \bar{\beta}_{(i,s)} = \tilde{\beta}_{(i,s)} \qquad \bar{f}'_{(i,s)} = \tilde{f}'_{(i,s)}$$

and the first part of (*ii*) is true.

In a similar way as in Lemma 2 the different possible cases of (61) must be checked, here (61.2) is explicitly treated. We have

$$i \in N_1 \cap \overline{N}_3 \cap N_{\ominus} \underset{(29)}{\overset{(25)}{\Longrightarrow}} \tilde{\lambda}_i = \tilde{\alpha}_{(q,i)} > (\gamma_i + \delta_i)\, Pr\, (v_i = g_i)$$

thus

$$\tilde{\alpha}_{(q,i)} = \tilde{w}_i - \tilde{w}_q \geqslant 0 \tag{65}$$

from which follows

$$\bar{\alpha}_{(q,i)} = \max\, \{0, \bar{w}_i - \bar{w}_q\}$$

$$\underset{(33)}{=} \max\, \{0, \tilde{w}_i - \tilde{w}_q - (\delta_i + \gamma_i)\, Pr(\underset{\sim}{v_i} = g_i)\} \tag{66}$$

$$\underset{(65)}{=} \tilde{\alpha}_{(q,i)} \quad (\delta_i + \gamma_i)\, Pr\, (\underset{\sim}{v_i} = g_i) > 0$$

Now, $\quad i \in N_1 \cap \overline{N_3} \Rightarrow (q, i) \in A^{(g, h)} \cap A^{(\bar{g}, \bar{h})}$

$$\Rightarrow \bar{f}'_{(q,i)} \underset{(32)}{=} \tilde{f}'_{(q,i)} \underset{(65)}{=} c^{(g,h)1}_{(q,i)}$$

but $\quad i \in N_1 \cap \overline{N}_3 \cap N_{\ominus}$ gives

$$\bar{f}'_{(q,i)} = c^{(g,h)1}_{(q,i)} \underset{(17)}{=} g_i = v_{i\mathbf{x}_i} \underset{(30)}{=} v_{i(\bar{\mathbf{x}}_{i+1})} = \overline{h_i}$$

$$\underset{(17)}{=} c^{(\bar{g},\bar{h})2}_{(q,i)} > 0 \tag{67}$$

whereas $c^{(g',h)1}_{(q,i)} = 0$ for $(q, \bar{i}) \in \overline{N_3}$.

(66), (67) show that (q, i) is out-of-kilter.

A discussion of the remaining cases of (61) is similar and omitted. □

PROOF OF LEMMA 4:

Note that f' is capacity-feasible. Now, because $a \in A \cup A_0$ is in kilter one has, see (27)

$$\alpha_a = \beta_a = 0 \quad \text{for} \quad a \in A(C) \cap (A \cup A_0) \tag{68}$$

to allow arc-flow alteration, and from this

$$\left.\begin{array}{ll} w_{I^1(a)} - w_{I^2(a)} = k_a^{(g,h)} & a \in A(C) \cap A \\ & \text{if} \\ w_q = w_s & (s, q) \in A(C) \end{array}\right. \tag{69}$$

additionally, $k_a^{(g,h)} = 0$ for $a \notin A$ from (19).

As out-of-kilter arcs belong to $\bigcup_{l=1}^{3} A_l$, at least one arc of the form (q, i) or (i, s), $i \in N$, belongs to $A(C)$, on the other hand

$$A(C) \cap A \neq \phi.$$

Assume without loss of generality that $N(C)$, the node sequence of C, is given by

$$N(C) = (q, i_1, \ldots\ldots, i_\rho, q),$$

(q, i_1), at least, is out-of-kilter, and $(q, i_1) \in A^+(C)$

C contains no cycle.

Now, $s \in \{i_2, \ldots, i_{\rho-1}\}$ or $s = i_\rho$ or $s \notin \{i_2, \ldots, i_\rho\}$.

Assume $s = i_\mu \in \{i_2, \ldots, i_{\rho-1}\}$ then

$$(i_{\mu-1}, i_\mu) \in A^+(C) \Rightarrow \alpha_{(i_{\mu-1}, i_\mu)} = 0$$
$$\Rightarrow \beta_{(i_{\mu-1}, i_\mu)} = w_{i_{\mu-1}} - w_s \geqq 0 \tag{70}$$

$$(i_{\mu+1}, i_{\mu}) \in A^{-}(C) \Rightarrow \beta_{(i_{\mu+1}, i_{\mu})} = 0$$

$$\Rightarrow \alpha_{(i_{\mu+1}, i_{\mu})} = w_s - w_{i_{\mu+1}} \geqq 0 \quad (71)$$

and $(q, i_1) \in A^{+}(C)$ (and out-of kilter)

$$\Rightarrow f'_{(q, i_1)} < c^{(q, h)2}_{(q, i_1)},$$

$$\beta_{(q, i_1)} = w_q - w_{i_1} > 0 \quad (72)$$

Now $(q, i_\rho) \in A^{-}(C) \Rightarrow f'_{(q, i_\rho)} > c^{(q, h)1}_{(q, i_\rho)}, \beta_{(q, i_\rho)} = 0,$

$$\alpha_{(q, i_\rho)} = w_{i_\rho} - w_q \geqq 0 \quad (73)$$

yields

$$\sum_{a \in A} k_a^{(q, h)} (fo)_a = (w_{i_1} - w_{i_{\mu-1}}) \triangle + (w_{i_{\mu+1}} - w_{i_\rho}) \triangle$$

$$\underset{(70),\,(71)}{\leqslant} (w_{i_1} - w_{i_\rho}) \,\Delta \underset{(72),\,(73)}{<} 0.$$

The remaining cases $s = i_\rho$ (note that now $(s, q) \in A^{+}(C)$ and use (69)) and $s \notin \{i_2, \ldots, i_\rho\}$ are easier to prove and omitted. □

PROOF OF THEOREM 2 :

As $\hat{f}' \neq \bar{f}'$ a flow change is yielded by successful application of, say $t \geqq 1$, primal (flow altering) phases of the out-of-kilter algorithm. Defining

$f'o_\nu : A^{(q, h)} \rightarrow \mathbf{R}$ with

$$(f'o_\nu)_a = \begin{cases} +\Delta_\nu, & a \in A^{+}(C_\nu) \\ -\Delta_\nu, & a \in A^{-}(C_\nu), \ \nu = 1, \ldots, t \\ 0, & a \notin A(C_\nu) \end{cases} \quad (74)$$

where $(A^{+}(C_\nu), A^{-}(C_\nu))$ is the partition of the arc set $A(C_\nu)$ of cycle C_ν according to the cycle-flow direction one has

$$\hat{f}' - \bar{f}' = \sum_{\nu=1}^{t} f'_{c_\nu}$$

(see *e.g.* [8] for the use of cycle-and cocycle-representations). Now, for

$$f'(0) = \bar{f}', \; w_i(0) = \bar{w}_i, \; i \in N^{(\bar{g}, \bar{h})},$$

the assumptions of Lemma 4 are fulfilled, see Lemmas 2 and 3.

But in-kilter arcs remain in-kilter and capacity-feasible circulations remain capacity-feasible during the out-of--kilter algorithm, so for

$$f'(\tau) = \bar{f}' + \sum_{\nu=1}^{\tau} f'_{c_\nu}$$

$$\text{and} \qquad w(\tau)_i, \; i \in N^{(\bar{g}, \bar{h})}, \; \tau = 0, \ldots, t \tag{75}$$

the assumptions of Lemma 4 are fulfilled, and $(k_a^{(\bar{g}, \bar{h})} = 0, \; a \notin A)$

$$\sum_{a \in A} k_a^{(\bar{g}, \bar{h})} \hat{f}'_a = \sum_{a \in A} k_a^{(\bar{g}, \bar{h})} f'(t)$$

$$= \sum_{a \in A} k_a^{(\bar{g}, \bar{h})} f'_a(t-1) + \sum_{a \in A} k_a^{(\bar{g}, \bar{h})} (f'_{c_t})_a$$

$$< \sum_{a \in A} k_a^{(\bar{g}, \bar{h})} f'_a(t-1) < \ldots < \sum_{a \in A} k_a^{(\bar{g}, \bar{h})} f'_a(0)$$

$$= \sum_{a \in A} k_a^{(\bar{g}, \bar{h})} \bar{f}'_a .$$

Now, (34) being valid one gets from Lemma 1, using the fact that

$$\tilde{f}'/A = \tilde{f} = f = \bar{f}'/A \qquad \text{(see (32))}$$

$$\sum_{a\in A} k_a \tilde{f}_a + \sum_{i\in N} E(\varphi_i^{\tilde{f}}) = \sum_{a\in A} k_a \bar{f}_a + \sum_{i\in N} E(\varphi_i^{\bar{f}})$$

$$= \sum_{a\in A} k_a^{(\bar{g},\bar{h})} f'_a + B^{(\bar{g},\bar{h})}$$

$$\underset{(34)}{>} \sum_{a\in A} k_a^{(\bar{g},\bar{h})} \hat{f}'_a + B^{(\bar{g},\bar{h})}$$

$$= \sum_{a\in A} k_a \hat{f}_a + \sum_{i\in N} E(\varphi_i^{\hat{f}})$$

(where $\hat{f} = \hat{f}'/_A$ is the "quasi"-flow corresponding to $\hat{f}'$) showing that $\tilde{f}$ is not optimal in (8). This proves Theorem 2. □

PROOF OF LEMMA 5 :

Notice, that (48) is true by (33), (52) when the modified version is started and remains trivially satisfied if

$$0 = c_{(s,q)}^{(\bar{g},\bar{h})1} < \bar{f}'_{(s,q)} < c_{(s,q)}^{(\bar{g},\bar{h})2} \tag{76}$$

as (s, q) is in-kilter arc, and because

$$\bar{f}'_{(s,q)} = c_{(s,q)}^{(\bar{g},\bar{h})2}$$

is impossible (see the proof of Theorem 1), the only case to discuss is

$$\bar{f}'_{(s,q)} = c_{(s,q)}^{(\bar{g},\bar{h})1} = 0$$

which is only possible if $N_3 = \overline{N_3} = N$.

Then, from Lemma 3 it remains to check (61·7) but

$$j \in N_3 \cap \overline{N}_3 \cap N_{\oplus} \text{ implies that}$$

$$\bar{f}'_{(q,j)} = c^{(\bar{g},\bar{h})1}_{(q,j)} < c^{(\bar{g},\bar{h})2}_{(q,j)}, \quad \bar{\beta}_{(q,j)} > 0 \tag{77}$$

(see also the proof of Lemma 2) and selecting (q, j) as starting out-of-kilter arc and supposing that (48) is valid, forces $s \in \overline{Y}$ and, of course, $q \in \overline{Y}$ to avoid a circulation change because of (35). By similar arguments

$$j \in N_3 \cap \overline{N}_3 \cap N_{\ominus}$$

$$\Rightarrow \bar{f}_{(j,s)} = c^{(\bar{g},\bar{h})1}_{(j,s)} < c^{(\bar{g},\bar{h})2}_{(j,s)}, \quad \bar{\beta}_{(j,s)} > 0 \tag{78}$$

and selecting (j, s) as starting out-of-kilter arc and supposing that (48) is valid, forces $q \in \bar{X}$ and, of course, $s \in \bar{X}$ to avoid a circulation change because of (37).

To summarize, one gets in all cases when using the starting out-of-kilter arc (q, j) or (j, s)

$$\{q, s\} \subset \begin{cases} \overline{Y}, & j \in N_{\oplus} \\ \bar{X}, & j \in N_{\ominus} \end{cases} \tag{79}$$

and the dual phase according to (42) after updating according to (47) shows that (48) remains true during further steps of the modified version. □

Proof of Theorem 3 :

First, it is shown how a new graph $G^{(\bar{g},\bar{h})}$ is created and an up to now undetected circulation change possibillty is found.

Discussion is restricted to the more complicated case $N_3 = \overline{N}_3 = N$ (because otherwise (76) holds and (s, q) can be used in either direction according to kilter-rules). Take, see (77) (case (78) is similar and omitted),

$$j \in N_3 \cap \overline{N}_3 \cap N_{\oplus}$$

and (q, j) as starting out-of-kilter arc. If, see (46),

$$0=\theta^*=\theta_0<\theta_{(\bar{X},\, \bar{Y})} \tag{80}$$

there exists $l \in N_0 \cap \bar{X}$ with, see (36),

$$\bar{\lambda}_l=(\gamma_l+\delta_l)Pr(\underset{\sim}{v_l}=\bar{g}_l) \quad \text{and} \quad \bar{\mu}_l=0.$$

Now, because of $N_3=\overline{N_3}=N$ one has $l \in N_3$, and from (79) $\{q, s\} \subset \bar{Y}$ and $(l, s) \in (\bar{X}, \bar{Y})$ is in-kilter arc with

$$\bar{\lambda}_l=\bar{\beta}_{(l,\, s)}>0 \Rightarrow \bar{f}'_{(l,\, s)}=c^{(\bar{g},\, \bar{h})2}_{(l,\, s)}=-\bar{g}_l.$$

Define

$$\bar{\bar{x}}_i=\begin{cases} \bar{x}_i-1 & ,\ i=l \\ \bar{x}_i & ,\ \text{otherwise} \end{cases}$$

(for (78) one gets $\bar{\bar{x}}_i=\bar{x}_i+1$, if $i=l$, $=\bar{x}_i$, otherwise) and construct the new graph $G^{(\bar{\bar{g}},\, \bar{\bar{h}})}$. Choosing

$$\bar{\bar{f}}'_a=\begin{cases} \bar{f}'_a & ,\ a \in A^{(\bar{\bar{g}},\, \bar{\bar{h}})} \cap A^{(\bar{g},\, \bar{h})} \\ 0 & ;\ \text{otherwise} \end{cases}$$

it is easy to see (by arguments as used in the proof of Lemma 2) that $\bar{\bar{f}}'$ is capacity-feasible in $G^{(\bar{\bar{g}},\, \bar{\bar{h}})}$. Additionally, defining

$$\bar{\bar{w}}_i=\begin{cases} \bar{w}_i-(\gamma_i+\delta_i)Pr(\underset{\sim}{v_i}=\bar{g}_i) & ,\ i=l \\ \bar{w}_i & ,\ \text{otherwise} \end{cases}$$

(for (78) one gets $\bar{\bar{w}}_i=\bar{w}_i+(\gamma_i+\delta_i)Pr(\underset{\sim}{v_i}=\bar{h}_i)$, if $i=l$, $=\bar{w}_i$, otherwise)

it is easy to see (by arguments as in the proof of Lemma 3) that for $\bar{\bar{f}}'$, $\bar{\bar{w}}_i$, $i \in N^{(\bar{\bar{g}},\, \bar{\bar{h}})}$, kilter-states are unchanged for $a \in A^{(\bar{\bar{g}},\, \bar{\bar{h}})} \cap A^{(\bar{g},\, \bar{h})}$, thus e.g. all arcs $a \in A \cup A_0$ remain in-kilter which is needed for application of Lemma 4.

So, (q, j) is still out-of kilter but for (l, s), now one gets

$$\bar{\bar{\alpha}}_{(l,s)}=\bar{\bar{\beta}}_{(l,s)}=0,\ \bar{\bar{f}}'_{(l,s)}=c^{(\bar{\bar{g}},\bar{\bar{h}})1}_{(l,s)}=c^{(\bar{\bar{g}},\bar{\bar{h}})2}_{(l,s)}$$

showing that, remember also (48),

(q, j), P_{jl} given from the construction of $\bar{X}$ (see (35)), (l, s), (s, q)

describes a cycle for a circulation change in $G^{(\bar{\bar{g}},\bar{\bar{h}})}$ with respect to $\bar{\bar{f}}'$, $\bar{\bar{w}}_i$, $i \in N^{(\bar{\bar{g}},\bar{\bar{h}})}$, for which application of Lemma 4 yields an improved solution. Now, continuing with the standard version of the out-of-kilter algorithm with respect to $G^{(\bar{\bar{g}},\bar{\bar{h}})}$ leads to a situation as described in theorem 2 with "–" signs replaced by "=" signs. If

$$0<\theta^*=\theta_0<\theta_{(\bar{X},\bar{Y})} \tag{81}$$

then, first, a dual change according to (42) and updating according to (47) has to be performed but the starting out-of-kilter arc (q, j) according to (77) fulfills $(q, j) \in S_2$ and remains out-of-kilter arc also after the dual change because of (81). Then, the next dual phase of the modified version with the same starting out-of-kilter arc (q, j) leads to the situation (80).

Thus, whenever θ^* is determined by (46) a new graph $G^{(\bar{\bar{g}},\bar{\bar{h}})}$ is created and continuation with the standard version of the out-of-kilter algorithm is suggested because of simplification of the solution procedure. Otherwise, θ^* is *always* determined by (43) leading after a finite number of dual changes to an enlargement of N_0 according to (44), (45), but N_0 is a subset of the finite set of nodes of the graph. This, if (46) does not occur, the "modified version of the out-of-kilter algorithm" will terminate with $N_0=N$ and $\hat{\omega}_i$, $i \in N^{(\bar{g},\bar{h})}$, s. t. condition (28) of Theorem 1 is fulfilled. ⊡

5. EXAMPLE

To discuss and demonstrate the different possibilities which can appear in the suggested algorithm, an example of simplest form is taken where the underlying graph $G=(\{1, 2, 3\}, \{(1, 2), (2, 3), (3, 1), (3, 2)\})$ (the incidence-mapping is omitted) with its arc capacities and arc costs given by the 3-tuple (c^1_a, c^2_a, k_a) is shown in Fig. 1, the data of the random variables in Table 1.

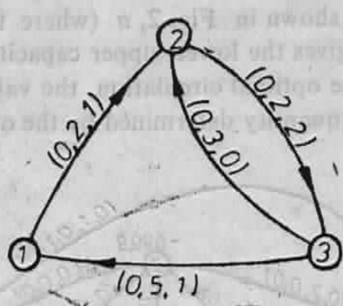

Fig. 1

node i	$(v_{ir}, p_{ir}),\ i=1, 2,\ldots, r_1$	v_{i0}	$v_{i(r_i+1)}$
1	$\left(-3, \frac{1}{10}\right), \left(-1, \frac{6}{10}\right), \left(1, \frac{1}{10}\right), \left(2, \frac{1}{10}\right)$ $\left(3, \frac{1}{10}\right)$	-20	20
2	$\left(-2, \frac{1}{4}\right), \left(0, \frac{1}{2}\right), \left(2, \frac{1}{4}\right)$	-20	20
3	$\left(0, \frac{1}{10}\right), \left(1, \frac{7}{10}\right), \left(2, \frac{1}{10}\right), \left(8, \frac{1}{10}\right)$	-20	20

Table 1

Fixing this situation the only data which remains to be specified are the cost-vectors for compensation γ, δ.

(I) $\qquad \delta=(0, 4, 1),\ \gamma=(1000, 0, 1)$

Starting with the capacity-feasible "quasi"-flow $f\equiv 0$, all $u^a_i(f)=0$, $i \in N$, and the following x_i values can be chosen as starting values

$$x=(2, 2, 1) \ \Rightarrow \ \begin{matrix} g=(-1, 0, 0) \\ (13) \quad h=(1, 2, 1) \end{matrix} \tag{82}$$

which gives (by (14))

$$N_1=N_2=\phi,\ N_3=\{1, 2, 3\} \tag{83}$$

and the graph $G^{(q,s)}$ shown in Fig. 2, *a* (where the 4-tuple assigned to each arc of the graph gives the lower, upper capacity, the arc cost term and the arc value of the optimal circulation, the value assigned to each node the optimal dual quantity determined by the out-of-kilter algorithm).

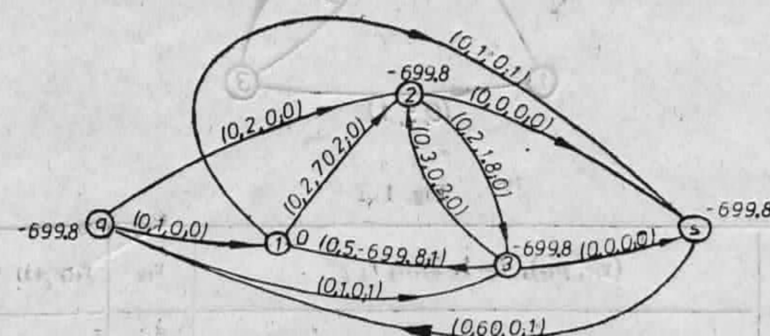

Fig. 2 *a*

In practice, when using an out-of-kilter subroutine working only for integer data, because of the rational probabilities multiplication and re-multiplication by an appropriate positive constant is necessary, and done here for the cost term. This gives the optimal circulation $\tilde{f}'$ and the dual quantities $\tilde{w}_i$, $i \in N \cup \{q, s\}$ with

$$\tilde{f}'_{(q,3)}=\tilde{f}'_{(3,1)}=\tilde{f}'_{(1,s)}=\tilde{f}'_{(s,q)}=1\ ;\ \tilde{f}'_{(i,j)}=0, \text{ else}$$

and

$$\tilde{w}_1=0,\ \tilde{w}_2=\tilde{w}_3=\tilde{w}_s=\tilde{w}_q=-699{\cdot}8$$

leading to (see (25), (26))

$$\tilde{\lambda}=(699.8, 0, 0)\ :\ \tilde{\mu}=(0, 0, 0)$$

and (see (29))

$$N_0=\{2, 3\},\ N_{\oplus}=\phi,\ N_{\theta}=\{1\}$$

because $\tilde{\lambda}_1=699.8>(\delta_1+\gamma_1)\cdot Pr(\underset{\sim}{v_1}=g_1)=(0+1000)\,\frac{6}{10}=600.$ Taking (see (30))

$$\mathbf{x}=(1,2,1) \underset{(13)}{\Longrightarrow} \begin{array}{l}\bar{g}=(-3,0,0)\\ \bar{h}=(-1,2,1)\end{array} \tag{84}$$

gives

$$\overline{N_1}=\phi,\ \overline{N_2}=\{1\},\ \overline{N_3}=\{2,3\} \tag{85}$$

and the new graph $G^{(\bar{g},\bar{h})}$ shown in Fig. 2, *b*.

Notice that $A^{(g,h)}\cap A^{(\bar{g},\bar{h})}=A^{(\bar{g},\bar{h})}$ thus, $\tilde{f}'\mid_{A^{(\bar{g},\bar{h})}}$ is a capacity-feasible circulation in $G^{(\bar{g},\bar{h})}$ according to Lemma 1 and will be used as starting circulation for the next application of the out-of-kilter algorithm (see (31)) together with the starting dual quantities (see (33))

$$\bar{w}_1=-99.8,\ \bar{w}_i=\tilde{w}_i,\ i\in\{2,3,q,s\}$$

giving

$$\hat{f}'_{(1,s)}=\hat{f}'_{(3,1)}=\hat{f}'_{(s,q)}=3,\ \hat{f}'_{(2,3)}=\hat{f}'_{(q,2)}=2,$$

$$\hat{f}'_{(q,3)}=1,\ \hat{f}'_{(i,j)}=0,\ \text{else}$$

and

$$\hat{w}_1=-601.8,\ \hat{w}_2=\hat{w}_s=\hat{w}_q=-699.8,\ \hat{w}_3=-701.6$$

as optimal quantities.

Because a circulation change occurred the procedure is restarted with $\bar{\mathbf{x}}$ instead of $\mathbf{x}$ according to theorem 2.

Continuing in the same way leads to the following complete sequence of $\mathbf{x}$-vectors

$$(2,2,1)\rightarrow(1,2,1)\rightarrow(1,2,2)\rightarrow(1,2,3). \tag{86}$$

The sequences of the underlying graphs and its optimal quantities which are used for the optimality checks and, if necessary, for the determination of the subsequent problems, are shown in Fig. 2.

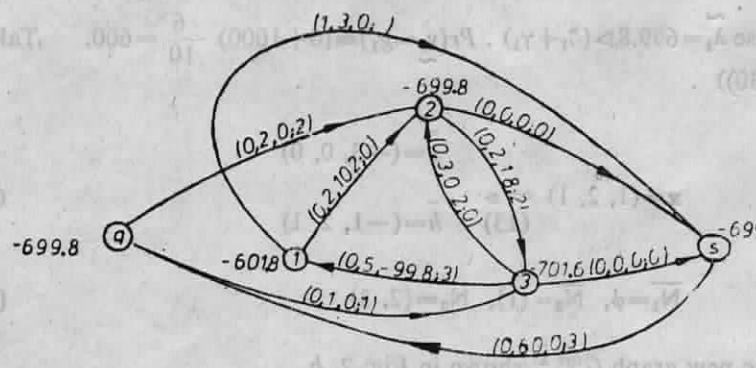

Fig. 2. b

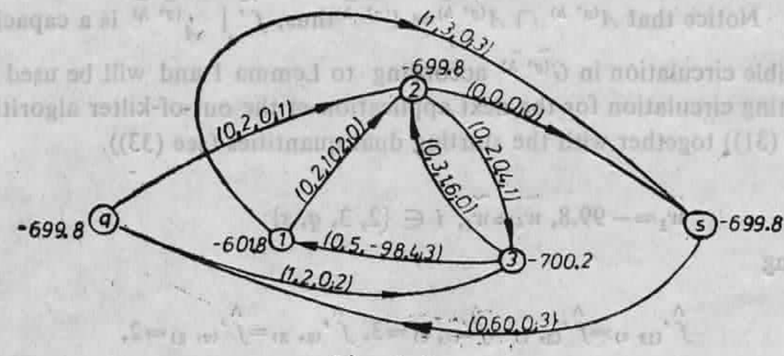

Fig. 2. c

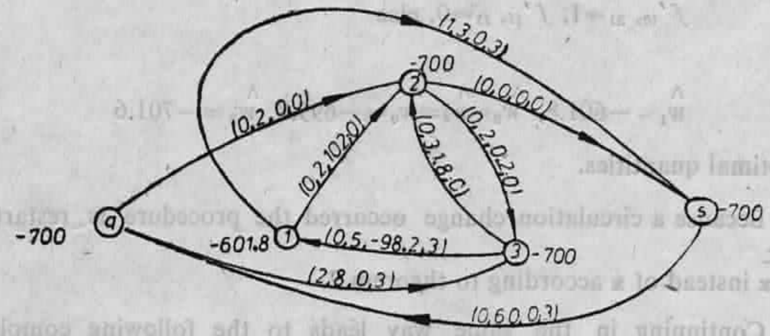

Fig. 2 d

Only for the last problem the details for the optimality check are explicitly given :

$$\mathbf{x} = (1, 2, 3)$$

$$w_1 = -601.8,\ w_2 = w_3 = w_q = w_s = -700 \Rightarrow \begin{array}{l} \lambda = (98.2, 0, 0) \\ \mu = (0, 0, 0) \end{array}$$

As $\lambda_i=0$, $i \in \{2, 3\}$ and $\mu_i=0$, $i \in \{1, 2, 3\}$ trivially satisfies the optimality condition (28) (see 7, (9.2)) and

$$\lambda_1=98.2 \leqq (\gamma_1+\delta_1) \cdot Pr(\underset{\sim}{v_1}=g_1)=(0+1000) \cdot Pr(\underset{\sim}{v_i}=v_{1x_1})$$

$$=1000 \cdot (0.1)=100$$

the "quasi"-flow

$$f_{(3,\,1)}=3\ ;\ f_{(1,\,2)}=f_{(2,\,3)}=f_{(3,\,2)}=0$$

is an optimal solution of the stochastic flow problem.

(II) $\delta=(0, 4, 1)$, $\gamma=(10, 0, 1)$

Starting with the capacity-feasible "quasi"-flow $f\equiv 0$, all $u_i^a(f)=0$, $i \in N$, and the following $\mathbf{x}_i$-values can be chosen as starting values (see (82))

$$\mathbf{x}=(2, 2, 1) \underset{(13)}{\Rightarrow} \begin{array}{l} g=(-1, 0, 0) \\ h=(1, 2, 1) \end{array} \tag{87}$$

and the graph $G^{(g,\,h)}$ is shown in Fig. 3. *a*. Application of the out-of-kilter algorithm gives

$$\tilde{f}'_{(q,\,3)}=\tilde{f}'_{(3,\,1)}=\tilde{f}'_{(1,\,s)}=\tilde{f}'_{(s,\,q)}=1,\ \tilde{f}'_{(i,\,j)}=0, \text{ else} \tag{88}$$

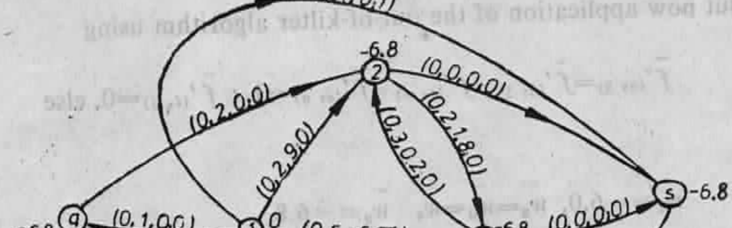

Fig. 3. a

and

$$\tilde{w}_1=0,\ \tilde{w}_2=\tilde{w}_3=\tilde{w}_q=\tilde{w}_s=-6.8 \tag{89}$$

leading to

$$\tilde{\lambda}=(6.8, 0, 0), \ \tilde{\mu}=(0, 0, 0)$$

and with

$$N_0=\{2, 3\}, \ N_{\oplus}=\phi, \ N_{\theta}=\{1\}$$

to the new **x**-vector

$$\bar{\mathbf{x}}=(1, 2, 1) \underset{(13)}{\Rightarrow} \begin{cases} \bar{g}=(-3, 0, 0) \\ \bar{h}=(-1, 2, 1) \end{cases}$$

and the new graph $G^{(g, h)}$ shown in Fig. 3, *b*.

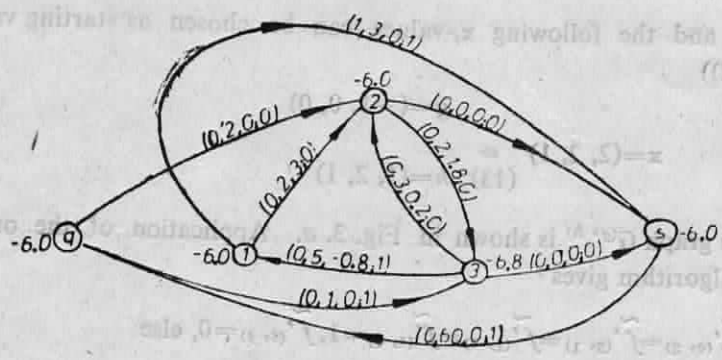

Fig. 3, b

But now application of the out-of-kilter algorithm using

$$\bar{f}'_{(q, 3)}=\bar{f}'_{(3, 1)}=\bar{f}'_{(1, s)}=\bar{f}'_{(s, q)}=1 \ ; \ \bar{f}'_{(i, j)}=0, \text{ else} \qquad (90)$$

and

$$\bar{w}_1=-6.0, \ \bar{w}_2=\bar{w}_3=\bar{w}_s \cdot \bar{w}_q=-6.8 \qquad (91)$$

as starting quantities, does not lead to a circulation change and in view of theorem 3 the modified version of the out-of-kilter algorithm should be applied.

Using again (90), (91) and $(1, s)$ as out-of-kilter arc gives (see (37), (38), because $1 \in N_{\theta}$)

$$\overline{Y}=\{1, 3\},\ \overline{X}=\{2, s, q\},\ \theta_0=(\delta_3+\gamma_3)\ .\ Pr(\tilde{v}_3=\bar{h}_3)=1.4$$

$$S_1=\{(2, 3)\},\ S_2=\{(1, s)\},\ \Theta_{(\overline{X}\ \overline{Y})}=\min\ \{1.8, 0.8\}=0.8$$

$$\theta^*=\min\ \{\theta_0,\ \theta_{(\overline{X},\ \overline{Y})}\}=0.8=\theta_{(\overline{X},\ \overline{Y})}<\theta_0 \qquad (92)$$

and according to (42)

$$w_i^*=\begin{cases}-6.0 & i\in\{2, s, q\}\\ -6.0 & i=1\\ -6.8 & i=3\end{cases}$$

Because

$$\alpha^*_{1s}=\beta^*_{1s}=0$$

the arc (1, s) is now in-kilter and (see (45))

$$N_0=\{1, 2, 3\}$$

thus the "quasi"-flow corresponding to the circulation given in (90) is an optimal solution of the stochastic flow problem. In this case, in view of (92) the standard version of the out-of-kilter algorithm would also lead to the optimal solution. This argument was also true for all used test-problems.

Thus, the authors c njecture that in practice application of a standard out-of-kilter subroutine is satisfactory for finding an optimal solution of the stochastic flow problem (by solving a *finite* number of minimnm cost circulation problems) a result similar to the situation of degeneracy in linear programming, where from a theoretical point of view the "modified" version is necessary to ensure finiteness as seen in the proofs.

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