Math. Operationsforsch. Statist., Ser. Optimization, Vol. 13 (1982) No. 3, 449–468

Project Scheduling via Stochastic Programming[1]

H. J. Cleef[2], W. Gaul[3]

Dedicated to Prof. Dr. W. Vogel, University of Bonn

Summary: If for a project (described by a non-empty set of activities, a relation on this set of activities the transitive closure of which is a strict order, and activity-completion-times assigned to the single activities) the activity-completion-times are assumed to be random variables a two-stage stochastic programming approach can be used for a cost-oriented project scheduling model. Completion-time estimates for the activity-completion-times are computed in such a way that, in order to meet a prescribed time-constraint for the project-completion-time, the expected costs for performing the activities according to the computed time-schedule are minimized. An example is included for illustration.

1. Introduction

Let be $A=\{a_1, \ldots, a_n\}$ a finite non-empty set, $0 \subset A \times A$ a binary ordering relation the transitive closure of which is a strict order (irreflexive, asymmetric, transitive), $Y_a \in \mathbb{R}_+$, $a \in A$. The tupel

$$(A, 0, (Y_a, a \in A))$$

(interpreting A as a set of activities for which, due to technological constraints, an order according to 0 has to be observed, and denoting by Y_a the activity-completion-time of $a \in A$) describes the usual information available in situations when a time-schedule for the performance of the given set of activities is needed. For the coordination and supervision of the activities graphtheoretical considerations have proved useful, see e. g. Ford/Fulkerson (1962), Fulkerson (1961), (1962), (1964), Golenko (1972) and are adopted here (see e. g. Kaerkes/Möhring (1978) for a description emphazising on order relations).

If the activity-completion-times are random variables the project-completion-time is a random variable the distribution function of which is difficult to obtain. Thus, efforts have been made to determine bounds for the expected project-completion-time, see e. g. Fulkerson (1962), Gaul (1981), Golenko (1972), and bounding distribution functions for the distribution function of the project-

1 This research has been supported by Sonderforschungsbereich 72, University of Bonn

2 Kienzle Apparate, Abt. 9.13, D - 7730 Villingen, BRD

3 Universität Karlsruhe (TH), Institut für Entscheidungstheorie und Unternehmensforschung, D - 7500 Karlsruhe 1, BRD

completion-time, see e. g. KLEINDORFER (1971), SHOGAN (1977). In VAN SLYKE (1963) one of the first attempts to apply Monte-Carlo methods was formulated.

As project-scheduling can be described by linear programming a first approach using tools from stochastic programming was given by CHARNES/COOPER/THOMPSON (1964) within a "chance-constrained model". The "distribution model" of stochastic programming, see e. g. KALL (1976), just corresponds to the problem of determining the distribtion function of the project-completion-time and shows the great computational difficulties imposed by numerical quadrature. The remaining "two-stage-model" of stochastic programming is used in this paper for a new approach to stochastic project scheduling emphazising the cost viewpoint arising within each planning situation. Whereas standard solution procedures for general two-stage programming problems even with restriction to two-stage programming with simple recourse and finite discrete random variables – as is assumed here – lead to computational difficulties, at least when using approximation arguments, see KALL (1974), the main result of this paper is to give a procedure which takes into account the special structure of the stochastic project scheduling problem and avoids these difficulties. This is done by constructing a finite sequence of non-stochastic flow problems the dimension of each of which is independent of the number of the realizations of the finite discrete random variables. Of course, the number of subproblems in the sequence can increase if, using approximation arguments, the number of realizations is increased but a modified "out-of-kilter" subroutine ensures that an improvement is yielded step by step. Thus, the algorithm is well-suited for approximation considerations e. g. when using empirical distribution functions in situations when the actual distribution functions are unknown. Similar considerations are obtained in CLEEF (1981) for the general linear case with simple recourse.

For an introduction to stochastic programming see e. g. KALL (1976), and for an extensive bibliography on papers concerning various topics of stochastic programming STANCU-MINASIAN/WETS (1976), computational aspects for two-stage stochastic programming problems are handled e. g. in KALL (1974), (1979), WETS (1974), (1975), the necessary graphtheoretical tools can be found in FORD/FULKERSON (1962), HARARY/NORMAN/CARTWRIGHT (1965), VOGEL (1967), only some basic formulations which relate particulary to this paper are given in section 2. Section 3 shows how optimal solutions of the subproblems are recognized to be also optimal for the main problem. The subproblems, which are special types of flow problems, can be solved by the "out-of-kilter" algorithm but to ensure finiteness of the sequence of subproblems a modified version of the "out-of-kilter" algorithm is needed. This modification and considerations concerning the construction of the sequence of subproblems are described in section 4. Section 5 deals with a numerical example.

2. Formulation of the Problem

Let

$$D_{s,t} = (V, X, f, (Y_x, x \in X))$$

(as graphtheoretical representation of $(A, 0, (Y_a, a \in A))$ where A corresponds with X, at least after the use of dummy arcs) denote the underlying stochastic project digraph which means that (V, X, f) is a *finite, acyclic, directed graph* with vertex set $V \neq \varnothing$, arc set X, incidence mapping $f = (f^1, f^2)$ with $f^i : X \to V$, $i = 1, 2$ ($f^1(x), f^2(x)$ denote the starting-, end-vertex of $x \in X$), vertex-basis s, vertex-contrabasis t, see e. g. HARARY/NORMAN/CARTWRIGHT (1965) for the graphtheoretical notations.

$(Y_x, x \in X)$ is a *random vector defined on a given probability space* (Ω, F, P) the components of which describe the completion-times (under normal conditions) for the single activities represented by the arcs of the digraph with

$$\mathsf{P}(Y_x \geqq y_x^0) = 1$$

where $y_x^0 \geqq 0$ is the *lowest possible* (*crash*) *completion-time*, $x \in X$. Of course, this description includes the non-stochastic case where all Y_x have degenerate distributions at $y_x \geqq y_x^0$.

The non-stochastic situation was described by FULKERSON (1961) under the assumption that costs for finishing activity x in d_x units of time are given by the *linear cost-function* $c(d_x) = b_x - o_x d_x$, $d_x \in [y_x^0, y_x]$, where $b_x, o_x \geqq 0$ are known integers allowing for associated costs of needed resources (machines, material, staff, etc.). For varying project-completion-time constraint $\lambda \geqq 0$ FULKERSON determined *project cost curves* by minimizing the project costs $\sum_{x \in X} c(d_x)$ dependent on λ.

For the more general stochastic case let X_d, X_r be a partition of X denoting the set of arcs with deterministic or random activity-completion-times. Notice that $X_r = \varnothing$ describes the non-stochastic situation and X_d contains the dummy arcs x with $y_x^0 = y_x = 0 = b_x$. Now, the new approach to stochastic project scheduling via the "two-stage model" of stochastic programming can be formulated in the following way:

Let be $\lambda > 0$ a prescribed time-constraint for the project-completion-time fullfilling $\lambda \geqq \max_{x \in X} y_x^0$. For every $x \in X$ an appropriate time-intervall $[y_x^0, y_x]$ (see (7) for the choice of y_x for $x \in X_r$) and a cost-function

$$c(d_x) = b_x - o_x d_x, \quad d_x \in [y_x^0, y_x], \quad b_x \geqq 0, \quad \begin{array}{l} o_x \geqq 0, \quad x \in X_d \\ o_x \text{ unrestricted}, \quad x \in X_r, \end{array} \tag{1}$$

can be selected. For $x \in X_d$ the description coincides with the FULKERSON-approach, d_x gives the *actual activity-completion-time*, the costs $c(d_x)$ increase if a shorter completion-time is scheduled. For $x \in X_r$ the activity-completion-time has to be estimated, here $c(d_x)$ are the costs for providing for d_x units of time those resources which are assumed to be sufficient for performing activity x. d_x is called completion-time-estimate and has to be chosen before the realization $Y_x(\omega)$, $\omega \in \Omega$, is known. To compensate the nonconformity of d_x with the realization of the activity-completion-time additional penalty-costs have to be defined according to

$$\varphi_{d_x}(Y_x(\omega)) = \begin{cases} q_x^+ \ (Y_x(\omega) - d_x) & > \\ 0 & , \quad Y_x(\omega) = d_x, \quad x \in X_r \\ -q_x^- \ (X_x(\omega) - d_x) & < \end{cases} \tag{2}$$

where q_x^+, q_x^-, are given real numbers satisfying

$$-q_x^+ < q_x^- \leqq o_x, \quad x \in X_r, \tag{3}$$

with the interpretation for $o_x \leqq 0$ (as used in the last example) that supplementary resources are more expensive, costs of not needed resources are refunded (or not needed resources can be hired out elsewhere).

Assuming for $x \in X_r$ that Y_x has finite expectation, φ_{d_x} is a *random variable* on (Ω, F, P) with finite expectation, and with

$$\Phi_{d_x} = \begin{cases} \mathsf{E}\varphi_{d_x} + c(d_x), & x \in X_r \\ c(d_x), & x \in X_d \end{cases} \tag{4}$$

the "two-stage model"-approach for project scheduling can be formulated as

$$\sum_{x \in X} \Phi_{d_x} = \text{Min!}, \quad d_x + \pi_{f^1(x)} - \pi_{f^2(x)} \leqq 0, \quad x \in X, \tag{5}$$
$$-\pi_s + \pi_t \leqq \lambda, \quad y_x^0 \leqq d_x \leqq y_x, \quad x \in X,$$

where π_i, $i \in V$, are the (unknown) times when with respect to the chosen d_x all activities x with $f^2(x) = i$ are completed. If the activity-completion-times Y_x for $x \in X_r$ are not yet finite discrete random variables one can reduce problem (5) to the case where the distribution functions of Y_x have finite support by using well-known approximation results, KALL (1974).

Thus, for $x \in X_r$ let $\{y_x^k \mid k = 1, \ldots, r_x\}$ denote the realizations of Y_x with

$$0 \leqq y_x^0 < y_x^1 < y_x^2 < \ldots < y_x^{r_x}, \quad r_x \in N \tag{6}$$
$$\mathsf{P}(Y_x = y_x^k) = p_x(k) > 0, \quad k = 1, \ldots, r_x, \quad \sum_{k=1}^{r_x} p_x(k) = 1$$

additionally for computational convenience

$$y_x = \max \{\max_{x \in X_r} \{y_x^{r_x}\}, \lambda\} + 1 = y_x^{r_x+1}, \qquad x \in X_r \tag{7}$$
$$p_x(0) = 0, \quad p_x(r_x + 1) = 0.$$

Now, because of (6) the random part of the objective function of (5) can be written in the form

$$\sum_{x \in X_r} \sum_{k=1}^{r_k} p_x(k)\, \varphi_{d_x}(y_x^k)$$

and each $\varphi_{d_x}(y_x^k)$ can be realized as *optimal value* of the linear recourse program

$$\varphi_{d_x}(y_x^k) = \min \{q_x^+ u_x^+ + q_x^- u_x^- \mid u_x^+ - u_x^- = y_x^k - d_x, \ u_x^+, u_x^- \geqq 0\}.$$

Thus, one gets the following equivalent formulation for (5), called SPSP (stochastic project scheduling problem)

$$\sum_{x\in X_r}\left[\sum_{k=1}^{r_x} p_x(k)\,[q_x^+ u_x^+(k)+q_x^- u_x^-(k)]+c(d_x)\right]+\sum_{x\in X_d} c(d_x)=\text{Min!} \tag{8}$$
$$d_x+\pi_{f^1(x)}-\pi_{f^2(x)}\leqq 0\,,\quad x\in X\,,\quad -\pi_s+\pi_t\leqq\lambda$$
$$d_x+u_x^+(k)-u_x^-(k)=y_x^k\,,\quad x\in X_r,\ k=1,\ldots,r_x\,,\quad -d_x\leqq -y_x^0\,,\quad x\in X$$
$$d_x\leqq y_x\,,\quad x\in X\,,\quad u_x^+(k),\,u_x^-(k)\geqq 0\,,\quad x\in X_r,\ k=1,\ldots,r_x$$

and its dual

$$-\lambda v-\sum_{x\in X_r}\sum_{k=1}^{r_x} y_x^k\mu_x(k)+\sum_{x\in X}(y_x^0\mu_x^0-y_x\mu_x+b_x)=\text{Max!}$$
$$w_x+\sum_{k=1}^{r_x}\mu_x(k)-\mu_x^0+\mu_x=o_x\,,\quad x\in X_r,\quad w_x-\mu_x^0+\mu_x=o_x\,,\quad x\in X_d$$
$$\sum_{\{x\mid f^1(x)=i\}} w_x-\sum_{\{x\mid f^2(x)=i\}} w_x=\begin{cases} v, & i=s\\ 0, & i\neq s,t\\ -v, & i=t\end{cases} \tag{9}$$
$$-q_x^+p_x(k)\leqq\mu_x(k)\leqq +q_x^-p_x(k)\,,\quad x\in X_r,\ k=1,\ldots,r_x$$
$$w_x\geqq 0,\ \mu_x^0\geqq 0,\ \mu_x\geqq 0,\ x\in X,\ v\geqq 0\,.$$

(8) and (9) describe a typical planning situation. One has to fix d_x-values under cost-viewpoints where planned time-reductions alter costs according to $c(d_x)$ and nonconformity with the actual realizations of the random activity-completion-times yields additional compensation costs (gains), see (2), (3). The project-completion-time constraint λ is prescribed, although a project cost curve approach, see FULKERSON (1961) for a description of the non-stochastic situation, may be of interest. Unfortunately, only for small but unrealistic values of r_x these linear programs could be solved e. g. by the revised simplex method.

But using the underlying graph (V, X, f) as basis for a sequence of properly chosen flow problems the dimension of which does not depend on the number of realizations of Y_x, $x\in X_r$, an algorithm can be described which terminates after a finite number of applications of a "(modified) out-of-kilter"-algorithm. To ensure finiteness restriction to rational data is made for costs as well as for the values y_x^k and their probabilities $p_x(k)$, $x\in X_r$, $k=1,\ldots,r_x$.

3. Optimality Condition

To start with consider the underlying graph (V, X, f) and the problem

$$\sum_{x\in X}\gamma_x d_x=\text{Max!}\,,\quad d_x+\pi_{f^1(x)}-\pi_{f^2(x)}\leqq 0\,,\quad x\in X \tag{10}$$
$$-\pi_s+\pi_t\leqq\lambda\,,\quad d_x\leqq\beta_x\,,\quad x\in X\,,\quad -d_x\leqq-\alpha_x\,,\quad x\in X$$

where α_x, β_x, γ_x are lower, upper bounds and costs for d_x on the arcs of the graph.

Flow theory on graphs obviously plays a role when working with the dual of (10)

$$\lambda v+\sum_{x\in X}[\beta_x g_x-\alpha_x h_x]=\text{Min}!\,,\ w_x+g_x-h_x=\gamma_x\,,\quad x\in X$$

$$\sum_{\{x\mid f^1(x)=i\}} w_x-\sum_{\{x\mid f^2(x)=i\}} w_x=\begin{cases} v, & i=s\\ 0, & i\neq s,t\\ -v, & i=t\end{cases} \tag{11}$$

$$w_x,\,g_x,\,h_x\geqq 0,\ x\in X,\ v\geqq 0\,.$$

How (11) can be solved in terms of a minimum cost circulation problem is shown at the end of this section and in section 4.

The connection between (10) (and its dual (11)) and SPSP in its discrete form (8) is given by the following theorem. For its proof one needs.

Lemma 1: *For $x\in X_r$ select $s_x\in\{0,1,\ldots,r_x\}$, then Φ_{d_x} is linear within*

$$y_x^{s_x}\leqq d_x\leqq y_x^{s_x+1} \tag{12}$$

and given by $\Phi_{d_x}=b(s_x)-o(s_x)\cdot d_x$ with

$$b(s_x)=q_x^+\,\mathsf{E}\,Y_x-(q_x^++q_x^-)\left\langle\sum_{k=1}^{s_x}p_x(k)\,y_x^k\right\rangle_{s_x=0}+b_x \tag{13}$$

$$o(s_x)=q_x^+-(q_x^++q_x^-)\,\mathsf{P}\,(Y_x\leqq y_x^{s_x})+o_x\,. \tag{14}$$

Remark: To avoid tedious discussions of special cases when selecting $s_x\in\{0,1,\ldots,r_x\}$ the following notation is used

$$\langle\text{expression }(s_x)\rangle_{s_x\gtreqless n}=\begin{cases}0, & s_x\gtreqless n\\ \text{expression }(s_x), & \text{otherwise}\end{cases}$$

Now, for $x\in X_r$ select $s_x\in\{0,1,\ldots,r_x\}$ and set

$$\alpha_x=\begin{cases}y_x^{s_x}\\ y_x^0\end{cases},\quad \beta_x=\begin{cases}y_x^{s_x+1}\\ y_x\end{cases},\quad \gamma_x=\begin{cases}o(s_x), & x\in X_r\\ o_x, & x\in X_d\end{cases} \tag{15}$$

then

Theorem 1: *Let d_x^*, $x\in X$, π_i^*, $i\in V$, describe an optimal solution of* (10) *and w_x^*, g_x^*, h_x^*, $x\in X$, v^* describe an optimal solution of* (11).
If

$$\begin{aligned} g_x^*&\leqq(q_x^++q_x^-)\,p_x\,(s_x+1)\quad \textit{for all}\quad x\in X_r\\ h_x^*&\leqq(q_x^++q_x^-)\,p_x(s_x)\qquad\ \textit{for all}\quad x\in X_r\ \textit{with}\ s_x>0\end{aligned} \tag{16}$$

then d_x^*, $x\in X$, π_i^*, $i\in V$, describe an optimal solution for SPSP.

Proof: Set

$$\begin{aligned}(u_x^+(k))^*&=\max\{0,\,y_x^k-d_x^*\}\\ (u_x^-(k))^*&=\max\{0,\,d_x^*-y_x^k\}\end{aligned},\quad x\in X_r,\ k=1,\ldots,r_x \tag{17}$$

then

$$d_x^*,\ x\in X,\ (u_x^+(k))^*,\ (u_x^-(k)^*,\ k=1\,,\ldots,r_x,\ x\in X_r),\ \pi_i^*,\ i\in V\,, \tag{18}$$

describe a feasible solution of (8).

Set, for $x \in X_r$, according to the selected s_x and $k \in \{1, \ldots, r_x\}$

$$\mu_x^*(k) = \begin{cases} q_x^- p_x(k)\,, & k < s_x \\ q_x^- p_x(k) - h_x^*\,, & k = s_x \\ -q_x^+ p_x(k) + g_x^*\,, & k = s_x + 1 \\ -q_x^+ p_x(k)\,, & k > s_x + 1 \end{cases} \quad x \in X \tag{19}$$

then, because of (16), $\mu_x^*(k)$ satisfies the inequalities of (9).

Set

$$\mu_x^{0*} = \begin{cases} \langle h_x^* \rangle_{s_x > 0} \\ h_x^* \end{cases}, \quad \mu_x^* = \begin{cases} \langle g_x^* \rangle_{s_x < r_x}, & x \in X_r \\ g_x^*, & x \in X_d \end{cases} \tag{20}$$

then

$$w_x^*,\ \mu_x^{0*},\ \mu_x^*,\ x \in X,\ (\mu_x^*(k),\ k = 1, \ldots, r_x,\ x \in X_r),\ v^* \tag{21}$$

describe a feasible solution of (9) because with (11), (14), (15), (19), (20)

$$w_x^* + \sum_{k=1}^{r_x} \mu_x^*(k) - \mu_x^{0*} + \mu_x^* = w_x^* + \left\langle \sum_{k=1}^{s_x} q_x^- p_x(k) \right\rangle_{s_x = 0} - \langle h_x^* \rangle_{s_x = 0}$$

$$- \left\langle \sum_{k = s_x + 1}^{r_x} q_x^+ q_x(k) \right\rangle_{s_x = r_x} + \langle g_x^* \rangle_{s_x = r_x} - \langle h_x^* \rangle_{s_x > 0} + \langle g_x^* \rangle_{s_x < r_x}$$

$$= w_x^* - q_x^+ + (q_x^+ + q_x^-)\, \mathrm{P}\,(Y_x \leqq y_x^{s_x}) - h_x^* + g_x^* = o_x\,, \quad x \in X_r$$

$$w_x^* - \mu_x^{0*} + \mu_x^* = w_x^* - h_x^* + g_x^* = o_x\,, \qquad x \in X_d$$

and the remaining equalities coincide.

Now, in view of duality theory it suffices to show that for (18), (21) the corresponding values of the objective functions of (8) and (9) are equal. From lemma 1, (15) and (17) the objective function of (8) can be rewritten as

$$\sum_{x \in X_r} (b(s_x) - o(s_x)\, d_x^*) + \sum_{x \in X_d} c(d_x^*) = \sum_{x \in X_r} b(s_x) + \sum_{x \in X_d} b_x - \sum_{x \in X} \gamma_x d_x^*$$

$$= \sum_{v \in X_r} b(s_x) + \sum_{x \in X_d} b_x - \lambda v^* - \sum_{x \in X} (\beta_x g_x^* - \alpha_x h_x^*)$$

where the last equality follows from the duality of (10), (11) and the assumed optimality of d_x^*, $x \in X$, π_i^*, $i \in V$, for (10), and of w_x^*, g_x^*, h_x^*, $x \in X$, v^* for (11). On the other hand, using (19), (20) the objective function of (9) can be rewritten as

$$-\lambda v^* + \sum_{x \in X_r} (y_x^0 \langle h_x^* \rangle_{s_x > 0} - y_x \langle g_x^* \rangle_{s_x < r_x} + b_x) + \sum_{x \in X_d} (y_x^0 h_x^* - y_x g_x^* + b_x)$$

$$- \sum_{x \in X_r} \left(\left\langle \sum_{k=1}^{s_x} y_x^k q_x^- p_x(k) \right\rangle_{s_x = 0} - \langle y_x^{s_x} h_x^* \rangle_{s_x = 0} \right)$$

$$- \sum_{x \in X_r} \left(\left\langle \sum_{k = s_x + 1}^{r_x} - y_x^k q_x^+ p_x(k) \right\rangle_{s_x = r_x} + \langle y_x^{s_x + 1} g_x^* \rangle_{s_x = r_x} \right)$$

$$= -\lambda v^* + \sum_{x \in X_r} b(s_x) + \sum_{x \in X_d} b_x + \sum_{x \in X_d} (y_x^0 h_x^* - y_x g_x^*)$$

$$+ \sum_{x \in X_r} (y_x^0 \langle h_x^* \rangle_{s_x > 0} + \langle y_x^{s_x} h_x^* \rangle_{s_x = 0} - y_x \langle g_x^* \rangle_{s_x < r_x} - \langle y_x^{s_x + 1} g_x^* \rangle_{s_x = r_x})$$

and, with regard to (15) the theorem is proved. ■

Now, it is obvious that by selecting s_x, $x \in X_r$, and using (15) the feasible region of SPSP is reduced to the feasible region of the maximization problem (10). Assuming that optimal solutions of (10) and (11) are known, theorem 1 gives a sufficient condition for the case when an optimal solution of (10) is also optimal for (8). Thus, for handling SPSP a finite sequence of deterministic project scheduling problems of the form (10) will be constructed each of which can be solved by suitable network flow algorithms. Standard network theory is not directly applicable as the objective function of (11) is not linear with respect to the flow but it is easy to verify, see Ford/Fulkerson (1962, p. 155) for similar considerations that, using the (slightly)

$$\begin{aligned} &\textit{modified graph } (\tilde{V}, \tilde{X}, \tilde{f}) \\ &\tilde{V} = V \\ &\tilde{X} = \{x_1 \mid x \in X\} \cup \{x_2 \mid x \in X\} \cup \{x_0\} \\ &\tilde{f}(z) = \begin{cases} f(x), & z = x_k, \, x \in X, \, k = 1, 2 \\ (t, s), & z = x_0 \end{cases} \qquad z \in \tilde{X} \end{aligned} \tag{22}$$

and

$$c_z = \begin{cases} -\beta_x, & z = x_1, \quad x \in X \\ \lambda, & z = x_0 \\ -\alpha_x, & z = x_2, \quad x \in X \end{cases} \qquad z \in \tilde{X} \tag{23}$$

$$a_z = \begin{cases} \gamma_x, & z = x_1, \quad x \in X \\ +\infty, & \text{otherwise} \end{cases} \qquad z \in \tilde{X}$$

(notice, that $\gamma_x \geqq 0$ for all $x \in X$ because of (1), (3), (15)) the following circulation problem

$$\begin{aligned} &\sum_{z \in \tilde{X}} c_z w_z = \text{Min!} \\ &\sum_{\{z \mid \tilde{f}^1(z) = i\}} w_z - \sum_{\{z \mid \tilde{f}^2(z) = i\}} w_z = 0, \quad i \in \tilde{V} \\ &0 \leqq w_z \leqq a_z, \quad z \in \tilde{X} \end{aligned} \tag{25}$$

is equivalent to (11).

An efficient solution procedure for a circulation problem is the *"out-of-kilter" algorithm* which determines an optimal circulation together with certain optimal vertex values, see Ford/Fulkerson (1962, p. 164) for more details or the next section. But, if

$$\left.\begin{aligned} &w_z^*, \, z \in \tilde{X}, \text{ decribe an optimal circulation} \\ &\tau_i^*, \, i \in \tilde{V}, \text{ describe optimal vertex values} \end{aligned}\right\} \text{ for (25)}$$

then

$$\begin{aligned} &w_x^* = w_{x_1}^* + w_{x_2}^*, \quad g_x^* = \gamma_x - w_{x_1}^*, \\ &h_x^* = w_{x_2}^*, \quad v^* = w_{x_0}^* \end{aligned} \qquad x \in X \tag{26}$$

and

$$\begin{aligned} &\pi_i^* = -\tau_i^*, \quad i \in V (= \tilde{V}) \\ &d_x^* = \min\{\beta_x, \pi_{\tilde{f}^2(x)}^* - \pi_{\tilde{f}^1(x)}^*\}, \quad x \in X \end{aligned} \tag{27}$$

describe an optimal solution of (10), (11).

4. (M.o.o.k.)-Algorithm and Sequence of Circulation Problems

A modified version of the "out-of-kilter" algorithm is needed to ensure that the sequence of solutions of the circulation problems (25) contains an optimal solution for SPSP after a finite number of steps.

Choose an arbitrary circulation w_z, $z \in \hat{X}$, and arbitrary vertex values τ_i, $i \in \hat{V}$, for problem (25) and use the abbreviation

$$\bar{c}_z = c_z + \tau_{\hat{f}^1(z)} - \tau_{\hat{f}^2(z)}, \quad z \in \hat{X}. \tag{28}$$

Recall, that $z \in \hat{X}$ is called *in-kilter* (i. k.) if

$$\bar{c}_z \begin{Bmatrix} > 0 & w_z = 0 \\ < 0 \Rightarrow w_z = a_z \\ = 0 & 0 \leqq w_z \leqq a_z \end{Bmatrix} \quad z \in \hat{X} \tag{29}$$

otherwise out-of-kilter (o. o. k.).

Denote $\hat{X}_r = \{x_1 \mid x \in X_r\} \cup \{x_2 \mid x \in X_r\}$ and use, for $x \in X_r$, $s_x \in \{0, 1, \ldots, r_x\}$ the abbreviation

$$e_z(s_x) = \begin{cases} (q_x^+ + q_x^-)\, p_x(s_x + 1), & z = x_1 \\ (q_x^+ + q_x^-)\, p_x(s_x), & z = x_2 \end{cases} \quad z \in \hat{X}_r \tag{30}$$

(notice that because of (3), (6), (7) $e_{x_1}(r_x) = 0$, $e_{x_2}(0) = 0$, $e_z(s_x) > 0$ otherwise) and call an (i. k.)-arc $z \in \hat{X}_r$ *in-kilter-critical* (i. k. c.) if

$$\begin{aligned} & a_z - e_z \leqq w_z \leqq a_z, && z = x_1 \\ & 0 \leqq w_z \leqq e_z(s_x), && z = x_2, \quad s_x > 0 \quad z \in \hat{X}_r \\ & 0 \leqq w_z, && z = x_2, \quad s_x = 0 \end{aligned} \tag{31}$$

The known (o. o. k.)-algorithm consists of an initial phase, a labeling phase, a circulation-alterration phase and a vertex value-alteration phase and tries to alter the circulation and/or the vertex values in order to find a situation in which all arcs are (i. k.) with respect to the actual circulation and the actual vertex values.

Assume that $z^{(*)}$ is (o. o. k.)-arc for which a $w_z^{(*)}$– increase (or – decrease) according to (o. o. k.)-rules is possible. Set $\hat{f}^2(z^{(*)}) = m$, $\hat{f}^1(z^{(*)}) = n$ (or $\hat{f}^1(z^{(*)}) = m$, $\hat{f}^2(z^{(*)}) = n$) and label m.

The modified "out-of-kilter" (m. o. o. k.)-algorithm differs from the (o. o. k.)-algorithm only in the following

Modified circulation-alteration phase: n has been labeled, i. e. a cycle C with arc set $\hat{X}(C) = \hat{X}^+(C) \cup \hat{X}^-(C)$ has been found where $\hat{X}^+(C)$, $\hat{X}^-(C)$ denotes the set of arcs z for which a w_z-increase, -decrease of value Δ_n would be possible. Determine

$$\begin{aligned} \Delta_{①} &= \min \{w_{x_1} - \gamma_x + e_{x_1}(s_x) \mid x_1 \in \hat{X}^-(C) && \text{and (i. k. c.)}\} \\ \Delta_{②} &= \min \{e_{x_2}(s_x) - w_{x_2} \mid x_2 \in \hat{X}^+(C) && \text{and (i. k. c.)}, s_x > 0\} \\ \Delta &= \min \{\Delta_{①}, \Delta_{②}, \Delta_n\} \end{aligned} \tag{32}$$

If $\Delta > 0$, perform

$$w_z = \begin{cases} w_z + \Delta, & z \in \hat{X}^+(C) \\ w_z, & z \notin \hat{X}(C) \\ w_z - \Delta, & z \in \hat{X}^-(C) \end{cases} \tag{33}$$

If $z^{(*)}$ is still (o. o. k.) correct the label of m (by Δ), erase all other labels and go to the labeling phase otherwise to the initial phase (to search for remaining (o. o. k.)-arcs).

If $\Delta_{①} = 0$, there exists at least one (i. k. c.)-arc $\tilde{x}_1 \in \hat{X}^-(C)$ with $w_{\tilde{x}_1} = \gamma_{\tilde{x}} - e_{\tilde{x}_1}(s_{\tilde{x}})$ (Notice $s_{\tilde{x}} < r_{\tilde{x}}$ because $s_{\tilde{x}} = r_{\tilde{x}} \Rightarrow e_{\tilde{x}_1}(s_{\tilde{x}}) = 0 \Rightarrow w_{\tilde{x}_1} = \gamma_{\tilde{x}}$ but $\tilde{x}_1$ is (i. k.)-arc $\Rightarrow c_{\tilde{x}_1} = 0 \Rightarrow \tau_{j^1(\tilde{x}_1)} - \tau_{j^2(\tilde{x}_1)} = \beta_{\tilde{x}} = y_{\tilde{x}}^{r_{\tilde{x}}+1}$, a contradiction to (7), see (27)). Change

$$s_x = \begin{cases} s_x + 1, & x = \tilde{x} \\ s_x, & \text{otherwise} \end{cases} \tag{34}$$

and all values which depend on s_x, $x \in X_r$. Erase all labels except for m and go to the labeling phase.

If $\Delta_{②} = 0$, there exists at least one (i. k. c.)-arc $\tilde{\tilde{x}}_2 \in \hat{X}^+(C)$ with $w_{\tilde{\tilde{x}}_2} = e_{\tilde{\tilde{x}}_2}(s_{\tilde{\tilde{x}}})$ and $s_{\tilde{\tilde{x}}} > 0$. Change

$$s_x = \begin{cases} s_x - 1, & x = \tilde{\tilde{x}} \\ x_x, & \text{otherwise} \end{cases} \tag{35}$$

and all values which depend on s_x, $x \in X_r$. Alter

$$w_z = \begin{cases} 0, & z = \tilde{\tilde{x}}_2 \\ w_{\tilde{\tilde{x}}_1} + e_{\tilde{\tilde{x}}_2}(s_{\tilde{\tilde{x}}}), & z = \tilde{\tilde{x}}_1 \text{ (the arc parallel to } \tilde{\tilde{x}}_2) \\ w_z, & \text{otherwise}. \end{cases}$$

Erase all labels except for m and go to the labeling phase.

Notice, that (34), (35) cause a change to a new circulation problem. The (m. o. o. k.)-algorithm will only be applied in the following

standard situation (36)

w_z, $z \in \hat{X}$, describe a feasible circulation for (25).
τ_i, $i \in \hat{V}$, are selected in such a way that
if $\tilde{z} \in \hat{X}$ is (o. o. k.) then
either $\tilde{z} \in \{x_1 \mid x \in X\}$ and $\bar{c}_{\tilde{z}} < 0$
or $\tilde{z} \in \{x_2 \mid x \in X\}$ and $\bar{c}_{\tilde{z}} > 0$.

One gets

Lemma 2: *Assume standard situation with at least one (starting)* (*o. o. k.*)-*arc* $z^{(*)} \in \hat{X}_r$. *If by application of the* (*m. o. o. k.*)-*algorithm no vertex value-alteration phase is performed then circulation alterations are found s. t. all* (i. k. c.)-arcs *remain* (*i. k. c.*) *and the starting* (o. o. k)-arc *becomes* (i. k. c.) *where a change to a new problem* (25)′ *may occur.*

In the following, relations between successive circulation problems, denoted by (25), (25)′, have to be discussed. The ′-notation is used for all associated quantities of the new problem (25)′, for the old problem (25) remember that it suffices to select $s_x \in \{0, 1, \ldots, r_x\}$, $x \in X_r$, α_x, β_x, γ_x, $x \in X$ see (15), respectively a_z, c_z, $z \in \hat{X}$ see (23), (24), to specify the situation.

Let w_z^*, $z \in \hat{X}$, describe an optimal circulation of (25). In view of (26), (27), (30) the optimality condition of theorem 1 is rewritten as

$$\begin{aligned} w_{x_1}^* &\geqq \gamma_x - e_{x_1}(s_x)\,, \\ w_{x_2}^* &\leqq e_{x_2}(s_x)\,, \quad s_x > 0\,, \end{aligned} \qquad x \in X_r\,. \tag{37}$$

Assume that (37) fails to be satisfied.

Using the following partition of X_r

$$\begin{aligned} X_\oplus &= \{x \in X_r \mid 0 \leqq w_{x_1}^* < \gamma_x - e_{x_1}(s_x)\} \\ X_\ominus &= \{x \in X_r \mid w_{x_2} > e_{x_2}(s_x),\ s_x > 0\}, \quad X_{(\text{i.k.c.})} = X_r \setminus (X_\oplus \cup X_\ominus) \end{aligned} \tag{38}$$

(that (38) is indeed a partition of X_r follows from (3), (6), (23) together with $\alpha_x < \beta_x$, $x \in X_r$, and the fact that (25) is formulated as minimum-problem yielding $w_{x_2}^* > 0 \Rightarrow w_{x_1}^* = \gamma_x$) to define

$$s_x' = \begin{cases} s_x + 1, & x \in X_\oplus \\ s_x - 1, & x \in X_\ominus \\ s_x\,, & x \in X_{(\text{i.k.c.})} \end{cases} \tag{39}$$

and, according to (15)

$$\alpha_x' = \begin{cases} y_x^{s_x'} = \beta_x \\ y_x^{s_x'} \\ \alpha_x \end{cases}, \quad \beta_x' = \begin{cases} y_x^{s_x'+1}\,, & x \in X_\oplus \\ y_x^{s_x'+1} = \alpha_x\,, & x \in X_\ominus \\ \beta_x\,, & x \in X_{(\text{i.k.c.})} \cup X_d \end{cases} \tag{40}$$

$$\gamma_x' = \begin{cases} \gamma_x - e_{x_1}(s_x) & x \in X_\oplus \\ \gamma_x + e_{x_2}(s_x)\,, & x \in X_\ominus \\ \gamma_x\,, & x \in X_{(\text{i.k.c.})} \cup X_d \end{cases}$$

and, according to (23), (24)

$$a_z',\ c_z',\ z \in \hat{X}$$

gives the new circulation problem (25)$'$ (One has $s_x' \geqq 0$ by definition of $X_\ominus$, and $s_x' \leqq r_x$ because $s_x' > r_x \Rightarrow s_x = r_x$, $x \in X_\oplus \underset{(26)}{\Rightarrow} g_x^* > 0 \underset{(26)}{\Rightarrow} w_{x_1}^* < \gamma_x$, but x_1 (i. k.) arc $\Rightarrow \bar{c}_{x_1} \geqq 0 \underset{(27}{\Rightarrow}$ $d_x^* = \beta_x = y_x^{r_x+1}$, a contradiction to (7)).

With

$$w_z' = \begin{cases} w_{x_1}^* + e_{x_2}(s_x) & z = x_1 \\ w_{x_2}^* - e_{x_2}(s_x)\,, & z = x_2 \\ w_z^*\,, & \text{otherwise} \end{cases} \quad x \in X_\ominus \quad z \in \hat{X} \tag{41}$$

$$\tau_i' = \tau_i^*\,, \quad i \in \hat{V} \tag{42}$$

one gets.

Lemma 3: w_z', $z \in \hat{X}$, *describe a feasible circulation of* (25)$'$. w_z', $z \in \hat{X}$, τ_i', $i \in \hat{V}$, *establish standard situation with* (o. o. k.)-arc $z^{(*)} \in \hat{X}_r$ *for* (25)$'$.

Of course, the main interest consists in checking whether the determined optimal solution of (25) respectively the new feasible solution of (25)′ according to (41), (42) yield an improvement for SPSP.

With (26), (27) the knowledge of the optimal solution of (25) is equivalent to the knowledge of an optimal solution of (10) for which one can show

Theorem 2:

(a) d_x^*, $x\in X$, π_i^*, $i\in V$, *describe a feasible solution of* (10)′ *with*

$$\sum_{x\in X_r} b(s_x)-o(s_x)\,d_x^*=\sum_{x\in X_r} b(s_x')-o(s_x')\,d_x^* .$$

(b) *If* $\hat{X}_\oplus=\{x\in X_\oplus \mid \bar{c}_{x_1}>0\}\neq\varnothing$ *then*

$$d_x''=\begin{cases}\min\{\beta_x',\ \pi^*_{f^2(x)}-\pi^*_{f^1(x)}\}, & x\in\hat{X}_\oplus\\ d_x^*, & \textit{otherwise}\end{cases},\quad x\in X,$$

$$\pi_i^*,\ i\in V,$$

describe a feasible solution of (10)′ *with*

$$\sum_{x\in X}\gamma_x' d_x^*<\sum_{x\in X}\gamma_x' d_x'' .$$

Proof: Choose $x\in X_\oplus$ then $w_{x_1}^*<\gamma_x$, but x_1 is (i. k.) for (25) with respect to w_z^*, $z\in\tilde{X}$, τ_i^*, $i\in\tilde{V}$, therefore (see (15), (27), (29))

$$\pi^*_{f^2(x)}-\pi^*_{f^1(x)}\geqq\beta_x=\alpha_x'<\beta_x'\Rightarrow\alpha_x'=d_x^*=y_x^{s_x'}\ (=\beta_x)<\beta_x'$$

and (see (13), (14))

$$b(s_x)-b(s_x')=(q_x^++q_x^-)\,p_x(s_x')\,y_x^{s_x'} \tag{43}$$

$$\times(o(s_x)-o(s_x'))\cdot d_x^*=(q_x^++q_x^-)\,p_x(s_x')\,y_x^{s_x'}.$$

Choose $x\in X_\ominus$ then $w_{x_2}^*>0$, but x_2 is (i. k.) for (25) with respect to w_z^*, $z\in\tilde{X}$, τ_i^*, $i\in\tilde{V}$, therefore (see (15), (27), (29))

$$\pi^*_{f^2(x)}-\pi^*_{f^1(x)}=\alpha_x=\beta_x'>\alpha_x'\Rightarrow\alpha_x'<d_x^*=y_x^{s_x'+1}=y_x^{s_x}\ (=\pi^*_{f^2(x)}-\pi^*_{f^1(x)})=\beta_x'$$

and (see (13), (14))

$$b(s_x')-b(s_x)=(q_x^++q_x^-)\,p_x(s_x)\,y_x^{s_x} \tag{44}$$

$$\times(o(s_x')-o(s_x))\,d_x^*=(q_x^++q_x^-)\,p_x(s_x)\,y_x^{s_x}.$$

For $x\in X_{(\text{l.k.c.})}\cup X_d$ all considered quantities are unchanged. Thus, part one of (a) is checked, part two of (a) follows with (43), (44). To prove (b) consider (see also (15))

$$x\in X_\oplus\Rightarrow\gamma_x'=\gamma_x-e_{x_1}(s_x)>w_{x_1}^*\geqq 0\Rightarrow\gamma_x'>0\quad\text{for } x\in\hat{X}_\oplus$$

$$x\in\hat{X}_\oplus\Rightarrow\bar{c}_{x_1}>0\Rightarrow\pi^*_{f^2(x)}-\pi^*_{f^1(x)}>\beta_x\Rightarrow\beta_x=\alpha_x'<\beta_x'$$

thus,

$$\alpha_x'=\beta_x=d_x^*<d_x''=\min\{\beta_x',\ \pi^*_{f^2(x)}-\pi^*_{f^1(x)}\}\leqq\beta_x'\quad\text{for } x\in\hat{X}_\oplus$$

and

$$\sum_{x\in X}\gamma_x'(d_x''-d_x^*)=\sum_{x\in\hat{X}_\oplus}\gamma_x'(d_x''-d_x^*)>0 .\ \blacksquare$$

Thus, by theorem 2 (a) an optimal solution of the old problem (10) can be used for the new problem (10)′ and leaves the objective function of SPSP unchanged, theorem 2 (b) gives an easy check for improvements. The next theorem shows how optimality for SPSP is reached.

Notice, that whenever a new problem (25)′ is started it is in standard situation, (o. o. k.)-arcs $z^* \in \tilde{X}_r$ exist, see lemma 3, and it remains in standard situation as long as no vertex value-alteration phase is performed, see lemma 2. Thus, assume standard situation for (25)′ with respect to $\bar{w}_z$, $z \in \tilde{X}$, τ_i', $i \in \tilde{V}$, and take a starting (o. o. k.)-arc $z^* \in \tilde{X}_r$. Denote by L the set of labeled vertices needed for the first application of the vertex value-alteration phase. Define for $M, Q \subset \tilde{V}\ (=V)$

$$\langle M, Q \rangle = \{z \mid z \in \tilde{X},\ \tilde{f}^1(z) \in M,\ \tilde{f}^2(z) \in Q\}$$

and

$$[M, Q] = \{x \mid \in X \text{ with } \{x_1, x_2\} \subset \langle M, Q \rangle\}$$

then, using the abbreviations

$$B_1 = \{x \mid x \in X,\ f^1(x) \in L,\ f^2(x) \in \bar{L}\} = [L, \bar{L}]$$
$$B_{11} = \{x \in B_1 \mid \bar{c}'_{x_1} \leqq 0\},\ B_{12} = x \in B_1 \mid \bar{c}'_{x_1} > 0\}$$
$$B_2 = \{x \mid x \in X,\ f^1(x) \in \bar{L},\ f^2(x) \in L\} = [\bar{L}, L]$$
$$B_{21} = \{x \in B_2 \mid \bar{c}'_{x_1} \geqq 0\},\ B_{22} = \{x \in B_2 \mid \bar{c}'_{x_1} < 0\}$$

one gets .

Lemma 4: (a) $x_0 \in \langle \bar{L}, L \rangle \Rightarrow \bar{w}_{x_0} = 0$, (b$_1$) $x \in B_{11} \Rightarrow \bar{w}_{x_1} = \gamma_x'\ (\bar{w}_{x_2} \geqq 0)$

(b$_2$) $x \in B_{21} \Rightarrow \bar{w}_{x_1} = 0,\ \bar{w}_{x_2} = 0$, (b$_3$) $x \in B_{22} \Rightarrow (\bar{w}_{x_1} \leqq \gamma_x'),\ \bar{w}_{x_2} = 0.$

Theorem 3: *Let* $(d_x^*)'$, $x \in X$, $(\pi_i^*)'$, $i \in V$ (resp. d_x^*, $x \in X$, π_i^*, $i \in V$), *describe the optimal solution of* (10)′ *(resp. of* (10)) *determined by the* (m. o. o. k.)-*algorithm applied to* (25)′ *(resp. to* (25)) *where for* (25)′ *the starting solution was* w_z', $z \in \tilde{X}$, τ_i', $i \in \tilde{V}$, *of Lemma* 3.

If during the application of the (m. o. o. k.)-*algorithm to* (25)′

(a) *a vertex-value alteration phase was performed then* $(d_x^*)'$, $x \in X$, $(\pi_i^*)'$, $i \in V$, *describe a better solution of* SPSP *than* d_x^*, $x \in X$, π_i^*, $i \in V$,

(b) *no vertex-value alteration phase was performed then* $(d_x^*)'$, $x \in X$, $(\pi_i^*)'$, $i \in V$, *describe an optimal solution for* SPSP.

Proof: Assume that for $\bar{w}_z$, $i \in \tilde{X}$, τ_i', $i \in \tilde{V}$, problem (25)′ is still in standard situation, $x_1^{(*)} \in \tilde{X}_r$ is the actual starting (o. o. k.)-arc with $\bar{c}'_{x_1^{(*)}} < 0$ (i. e. $\bar{w}_{x_1^{(*)}} < \gamma'_{x^{(*)}}$) and L is the set of labeled nodes which, now, has to be used for the application of the vertex value-alteration phase for the first time. Let

$$\bar{\tau}_i = \begin{cases} \tau_i' + \vartheta\,, & i \in \bar{L} \\ \tau_i'\,, & i \in L \end{cases} \tag{45}$$

denote the new vertex values.

From the circulation conservation condition one has for $\bar{w}_z$, $z \in \tilde{X}$,

$$\sum_{z \in \langle L, \bar{L} \rangle} \bar{w}_z = \sum_{z \in \langle \bar{L}, L \rangle} \bar{w}_z\ . \tag{46}$$

and from lemma 4

$$\sum_{z \in \langle \bar{L}, L \rangle} \bar{w}_z \geqq \sum_{x \in B_1} (\bar{w}_{x_1} + \bar{w}_{x_2}) \geqq \sum_{x \in B_{11}} (\bar{w}_{x_1} + \bar{w}_{x_2}) \geqq \sum_{x \in B_{11}} \bar{w}_{x_1} = \sum_{x \in B_{11}} \gamma_x' \tag{47}$$

and

$$\sum_{z\in\langle L,\bar{L}\rangle} \bar{w}_z = \sum_{x\in B_2} (\bar{w}_{x_1} + \bar{w}_{x_2}) = \sum_{x\in B_{22}} (\bar{w}_{x_1} + \bar{w}_{x_2}) = \sum_{x\in B_{22}} \bar{w}_{x_1} < \sum_{x\in B_{22}} \gamma'_x \tag{48}$$

where the strict inequality is induced by the chosen actual starting (o. o. k.)-arc $x_1^{(*)}$.

Constructing for $x\in X$ with $f^1(x)=i$, $f^2(x)=j$ the new quantities $\bar{d}_x$, $x\in X$ by

$$\{i, j\}\subset L \text{ or } \{i, j\}\subset \bar{L} \Rightarrow d_x = d'_x$$

$$x\in B_{11} \Rightarrow \bar{\tau}_i - \bar{\tau}_j = \tau'_i - \tau'_j - \vartheta < \tau'_i - \tau'_{j_1} \leqq \beta'_x \Rightarrow \bar{d}_x = d'_x - \vartheta$$

$$x\in B_{12} \Rightarrow \bar{\tau}_i - \bar{\tau}_j = \tau'_i - \tau'_j - \vartheta \geqq \beta'_x \Rightarrow \bar{d}_x = d'_x$$

$$x\in B_{21} \Rightarrow \bar{\tau}_i - \bar{\tau}_j = \tau'_i + \vartheta - \tau'_j > \beta'_x \Rightarrow \bar{d}_x = d'_x$$

$$x\in B_{22} \Rightarrow \bar{\tau}_i - \bar{\tau}_j = \tau'_i + \vartheta - \tau'_j \leqq \beta'_x \Rightarrow \bar{d}_x = d'_x + \vartheta$$

yields

$$\sum_{x\in X} \gamma'_x (\bar{d}_x - d'_x) = \sum_{x\in B_{11}} (-\vartheta)\gamma'_x + \sum_{x\in B_{22}} \vartheta\gamma'_x = \Big(\sum_{x\in B_{22}} \gamma'_x - \sum_{x\in B_{11}} \gamma'_x\Big)\vartheta > 0$$

by (45), (46), (47), (48) and $\vartheta > 0$.

But $\bar{d}_x$, $x\in\hat{X}$, $\bar{\pi}_i = -\bar{\tau}_i$, $i\in\hat{V}$ is a feasible solution of (10)′ which proves (a). The proof of (b) follows by repeated application of lemma 2 unless all arcs of $\hat{X}_r$ are (i. k. c.) and, of course, all arcs of $\hat{X}\backslash\hat{X}_r$ are (i. k.) which means that the optimality condition of theorem 1 is fulfilled. ■

Now, the suggested solution procedure for project scheduling via stochastic programming can be summarized as follows:

First, find a feasible starting circulation problem of the form (25), e.g. check (by one of the known path algorithms) whether the prescribed time constraint λ is compatible with the lowest possible completion times y_x^0, $x\in X$, in which case the minimum cost circulation problem (25) (with $s_x\equiv 0$, $x\in X_r$) has feasible solutions and can be used as starting problem.

Second, a sequence of circulation problems of the form (25) is generated where a new problem is created as long as the optimality check of theorem 1 fails to be satisfied for the actual problem. Here, lemma 3 ensures that at the beginning each problem is in standard situation and has (o. o. k.)-arcs.

Third, the difficulty which arises when the value of the objective function of the stochastic project scheduling problem determined from the optimal solutions of successive circulation problems (25), (25)′ remains unchanged is handled by application of the (m.o.o.k.)-algorithm. The modification ensures that either an improvement for the objective function value or optimality is yielded.

Fourth, this monotony-argument and the assumed rationality of the data cause the finiteness of the procedure.

5. Example

To discuss and demonstrate the suggested algorithm an example of simplest form is taken where the project is described by the set of activities $A=\{a_1, \ldots, a_5\}$, the random activity-completion-times Y_a, $a\in A$, the relizations of which together with the costs for compensation are given by Tab. 1, and the ordering relation $0\subset A\times A$ which is represented by the arc-adjacency relation of the graph $D_{1,4}$ shown in Fig. 1 where the activities correspond with the arcs of the graph as indicated.

To stress the stochastic aspects $b_{a_i}=o_{a_i}=0$, $a_i\in A$, are chosen in this example. Prescribing the project-completion-time constraint $\lambda=9$ yields the value $y_{a_i}=17$ for all $a\in A$ (see (7)).

Tab. 1

activity a_i	$y^k_{a_i}$, $k=1, 2, 3$	$p_{a_i}(k)$, $k=1, 2, 3$	$y^0_{a_i}$	b_{a_i}	o_{a_i}	$q^+_{a_i}$	$q^-_{a_i}$
a_1	3, 5, 7	0.25, 0.375, 0.375	1	0	0	4	−1
a_2	4, 6, 8	0.375, 0.125, 0.5	2	0	0	9	−1
a_3	4, 5, 6	0.75, 0.125, 0.125	3	0	0	5	−3
a_4	5, 8, 11	0.3, 0.4, 0.3	2	0	0	6	−5
a_5	6, 11, 16	0.125, 0.4, 0.475	1	0	0	10	−2

Using the graphtheoretical representation of Fig. 1 and fixing the starting quantity $s_{(i,j)}=0$ for each arc (i, j) of $D_{1,4}$ the optimal solution was determined in three iterations. In each iteration a minimum cost circulation problem has to be solved with respect to the graph of Fig. 2 having parallel arcs between nodes i, j denoted by $(i, j)_1$, $(i, j)_2$ and lower, upper arc-capacity–and arc-cost-terms as given in (22)–(25).

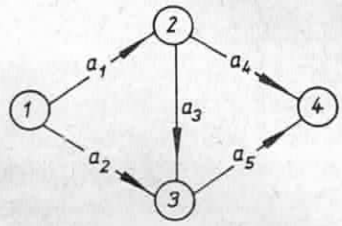

Fig. 1

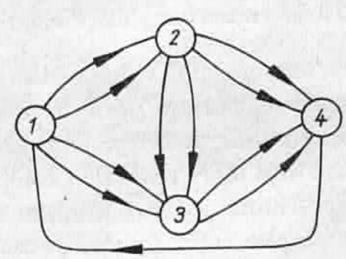

Fig. 2

All quantities of interest for the single iterations are listed below, for notation see also (26), (27) (In practise when using an "out-of-kilter" subroutine working only with integer data because of the rational probabilities multiplication and remultiplication by an appropriate positive constant is necessary and done here for the $\gamma_{(i,j)}$-terms).

Tab. 2

arc (i, j)	$s_{(i,j)}$	$\alpha_{(k,j)}$	$\beta_{(i,j)}$	$\gamma_{(i,j)}$	$w^*_{(i,j)_1}$	$w^*_{(i,j)_2}$	$g^*_{(i,j)}$	$e_{(i,j)_1}(s_{(i,j)})$	$h^*_{(i,j)}$	$e_{(i,j)_2}(s_{(i,j)})$	$d^*_{(i,j)}$
(1, 2)	0	1	3	4	4	6	0	0,75	6	0	1
(1, 3)	0	2	4	9	0	0	$\underline{9}$	$\underline{3}$	0	0	4
(2, 3)	0	3	4	5	5	5	0	1,5	5	0	3
(2, 4)	0	2	5	6	0	0	$\underline{6}$	$\underline{0,3}$	0	0	5
(3, 4)	0	1	6	10	10	10	0	1	0	0	5

$\pi^*_1=0 \quad \pi^*_2=1 \quad \pi^*_3=4 \quad \pi^*_4=9$ Iteration 1

Tab. 3

arc (i, j)	$s_{(i,j)}$	$\alpha_{(i,j)}$	$\beta_{(i,j)}$	$\gamma_{(i,j)}$	$w^*_{(i,j)_1}$	$w^*_{(i,j)_2}$	$g^*_{(i,j)}$	$e_{(i,j)_1}(s_{(i,j)})$	$h^*_{(i,j)}$	$e_{(i,j)_2}(s_{(i,j)})$	$d^*_{(i,j)}$
(1, 2)	0	1	3	4	4	4	0	0,75	4	0	1
(1, 3)	1	4	6	6	6	0	0	1	0	3	5
(2, 3)	0	3	4	5	4	0	1	1,5	0	0	4
(2, 4)	1	5	8	5,7	4	0	$\underline{1,7}$	$\underline{0,4}$	0	0,3	8
(3, 4)	0	1	6	10	10	0	0	1	0	0	4

$\pi^*_1=0 \quad \pi^*_2=1 \quad \pi^*_3=5 \quad \pi^*_4=9$ Iteration 2

Tab. 4

arc (i, j)	$s_{(i,j)}$	$\alpha_{(i,j)}$	$\beta_{(i,j)}$	$\gamma_{(i,j)}$	$w^*_{(i,j)_1}$	$w^*_{(i,j)_2}$	$g^*_{(i,j)}$	$e_{(i,j)_1}(s_{(i,j)})$	$h^*_{(i,j)}$	$e_{(i,j)_2}(s_{(i,j)})$	$d^*_{(i,j)}$
(1, 2)	0	1	3	4	4	5,3	0	0,75	5,3	0	1
(1, 3)	1	4	6	6	6	0	0	1	0	3	5
(2, 3)	0	3	4	5	4	0	1	1,5	0	0	4
(2, 4)	2	8	11	5,3	5,3	0	0	0,3	0	0,4	8
(3, 4)	0	1	6	10	10	0	0	1	0	0	4

$\pi^*_1=0 \quad \pi^*_2=1 \quad \pi^*_3=5 \quad \pi^*_4=9$ Iteration 3

Those values for which the optimality-check of theorem 1 fails to be satisfied are underlined.

The optimal value of the objective function is 131.275. The computed optimal time-schedule $d^*_{(i,j)}$ of iteration 3 gives $d^*_{(1,2)}=1=y^0_{(1,2)}$ indicating that the corresponding activity should be performed as quick as possible.

For a more realistic example consider the set of activities $A=\{a_1, \ldots, a_{25}\}$, the deterministic and random activity-completion-times Y_a, $a\in A$, which together with the costs for compensation are given by Tab. 5, and the ordering relation $0\subset A\times A$ represented by the arc-adjacency relation of the graph $D_{1,15}$ shown in Fig. 3 where the activities correspond with the arcs of the graph as indicated. Choosing the project-completion-time constraint $\lambda=30$ yields

$$y^6_{a_i}=31 \quad \text{for all} \quad a_i\in A\,.$$

Using again $s_{(i,j)}=0$ for each arc $(i,j)\in X_r$ of $D_{1,15}$ as starting quantities the optimal solution was now determined in 8 iterations. In view of space limitation

Tab. 5

activity a_i	$y^k_{a_i}$, $k=1, 2, \ldots, r_i$ (or y_{a_i} in det. case)	$p^k_{a_i}$, $k=1, 2, \ldots, r_i$	$y^0_{a_i}$	b_{a_i}	o_{a_i}	$q^+_{a_i}$	$q^-_{a_i}$
a_1	3, 5, 10, 13, 20	0.2, 0.3, 0.3, 0.15, 0.05	1	50	− 2	4	− 3
a_2	3, 5, 10, 13, 20	0.2, 0.3, 0.3, 0.15, 0.05	1	120	− 5	6	− 5
a_3	4, 6, 8, 10, 12	0.15, 0.25, 0.25, 0.2, 0.15	2	40	− 3	5	− 4
a_4	2, 3, 5, 6, 7	0.1, 0.2, 0.5, 0.1, 0.1	1	80	−10	12	−11
a_5	0, deterministic (dummy)		0	0	0	—	—
a_6	6, 9, 15, 20, 25	0.175, 0.55, 0.2 , 0.025, 0.05	3	300	− 9	12	−10
a_7	6, 7, 8, 12, 18	0.15, 0.075, 0.3 , 0.3, 0.175	2	155	− 7	9	− 7
a_8	6, 9, 15, 20, 25	0.175, 0.55, 0.2 , 0.025, 0.05	3	110	− 4	6	− 5
a_9	12, deterministic	—	6	100	8	—	—
a_{10}	2, 3, 5, 6, 7	0.1, 0.2, 0.5, 0.1, 0.1	1	10	− 1	6	− 2
a_{11}	13, deterministic	—	5	180	12	—	—
a_{12}	12, deterministic	—	3	180	14	—	—
a_{13}	3, 5, 10, 13, 20	0.2, 0.3, 0.3, 0.15, 0.05	1	60	− 3	14	− 4
a_{14}	24, deterministic	—	6	500	18	—	—
a_{15}	6, 9, 15, 20, 25	0.175, 0.55, 0.2, 0.025, 0.05	3	30	− 1	5	− 3
a_{16}	3, 5, 10, 13, 20	0.2, 0.3, 0.3, 0.15, 0.05	1	200	− 9	13	−10
a_{17}	18, deterministic	—	6	200	10	—	—
a_{18}	4, 6, 8, 10, 12	0.15, 0.25, 0.25, 0.2, 0.15	2	50	− 4	7	− 5
a_{19}	0, deterministic (dummy)	—	0	0	0	—	—
a_{20}	2, 3, 5, 6, 7	0.1, 0.2, 0.5, 0.1, 0.1	1	70	− 9	12	−10
a_{21}	6, 7, 8, 12, 18	0.175, 0.55, 0.2, 0.025, 0.05	2	200	−11	13	−12
a_{22}	6, 9, 15, 20, 25	0,175, 0.55, 0.2, 0.025, 0.05	3	80	− 3	6	− 4
a_{23}	2, 3, 5, 6, 7	0,1, 0.2, 0.5, 0.1, 0.1	1	40	− 5	7	− 6
a_{24}	4, 6, 8, 10, 12	0.15, 0.25, 0.25, 0.2, 0.15	2	20	1	3	− 2
a_{25}	6, 9, 15, 20, 25	0.175, 0.55, 0.2, 0.025, 0.05	3	30	1	3	− 2

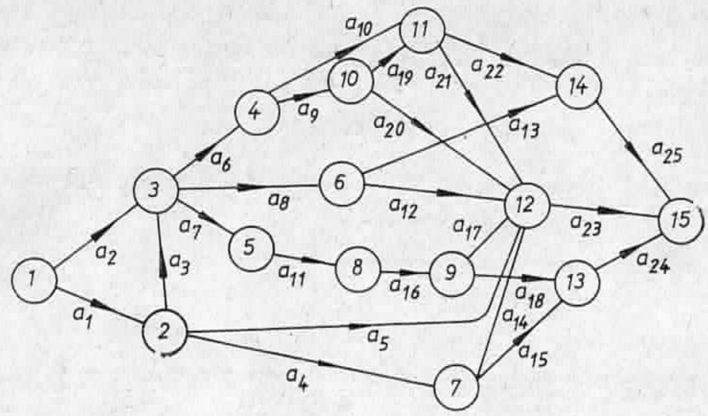

Fig. 3

only the generated sequences $\{s^n_{(i,j)}\}_{0 \leqq n \leqq 7}$ for the arcs (i, j) of $D_{1,15}$ are listed in Tab. 6 together with the optimal solution $d^*_{(i,j)}$ for (8) yielded from the last iteration.

The optimal value of the objective function is 2208.000, the optimal vertex

Tab. 6

$\mathrm{arc}_{(i,j)}$	$s^n_{(i,j)}$, $n = 0, \ldots, 7$	$d^*_{(i,j)}$
(1, 2)	0 ——— 0	1
(1, 3)	0 0 0 0 0 0 1 0	3
(2, 3)	0 ——— 0	2
(2, 7)	0 1 2 2 2 1 0 0	2
(2, 12)	0 deterministic	0
(3, 4)	0 1 1 1 0 — 0	3
(3, 5)	0 ——— 0	2
(3, 6)	0 1 2 ——— 2	9
(4, 10)	0 deterministic	12
(4, 11)	0 1 2 3 4 5 – 5	12
(5, 8)	0 deterministic	13
(6, 12)	0 deterministic	12
(6, 14)	0 1 2 3 4 — 4	15
(7, 12)	0 deterministic	24
(7, 13)	0 1 2 3 4 — 4	25
(8, 9)	0 0 ——— 0	1
(9, 12)	0 deterministic	10
(9, 13)	0 1 2 2 2 2 3 3	9
(10, 11)	0 deterministic	0
(10, 12)	0 1 2 3 4 5 5 5	11
(11, 12)	0 1 2 3 ——— 3	11
(11, 14)	0 1 ——— 1	9
(12, 15)	0 ——— 0	1
(13, 15)	0 1 1 1 1 0 0 0	2
(14, 15)	0 ——— 0	3

values are

$$\pi_1^*=0, \quad \pi_2^*=1, \quad \pi_3^*=3, \quad \pi_4^*=6, \quad \pi_5^*=5,$$
$$\pi_6^*=12, \quad \pi_7^*=3, \quad \pi_8^*=18, \quad \pi_9^*=19, \quad \pi_{10}^*=18,$$
$$\pi_{11}^*=18, \quad \pi_{12}^*=29, \quad \pi_{13}^*=28, \quad \pi_{14}^*=27, \quad \pi_{15}^*=30.$$

For the described examples a computer code using the (m.o.o.k.)-algorithm was developed. The number of needed iterations was always small compared with the total number of possible choices.

Acknowledgement

The authors would like to thank one the referees for his comments which improved an earlier version of the paper. This version [3] also contains the proofs of lemma 1–4.

References

[1] Charnes, A.; W. W. Cooper; G. L. Thompson: Critical Path Analysis via Chance-Contrained and Stochastic Programming. Operations Research **12** (1964) 460–470.
[2] Cleef, H. J.: A Solution Procedure for the Two-Stage Stochastic Program with Simple Recourse, Zeitschrift für Operations Research, **25** (1981) 1–13.
[3] Cleef, H. J.; W. Gaul: Project Scheduling via Stochastic Programming. SFB-72 Preprint 367, University of Bonn, 1980.
[4] Ford, L. R.; D. R. Fulkerson: Flows in Networks. Princeton University Press, 1962.
[5] Fulkerson, D. R.: A Network Computation for Project Cost Curves. Managem. Sc. **7** (1961) 167–178.
[6] Fulkerson, D. R.: Expected Critical Path Length in PERT Type Networks. Operations Research **10** (1962) 808–817.
[7] Fulkerson, D. R.: Scheduling in Project Networks. Proc. IBM Scient. Comp. Symp. Comb. Problems (1964) 73–92.
[8] Gaul, W.: Bounds for the Expected Duration of a Stochastic Project Planning Model. J. Infor. & Optimiz. Sc. **2** (1981) 45–63.
[9] Golenko, D. I.: Statistische Methoden der Netzplantechnik. Teubner Verlag, 1972.
[10] Harary, F.; F. Z. Norman; D. Cartwright: Structural Models: An Introduction to the Theory of Directed Graphs. Wiley, 1965.
[11] Kaerkes, R.; R. Möhring: Vorlesungen über Ordnungen und Netzplantheorie. Schriften zur Informatik, Angew. Mathematik 45, RWTH Aachen, 1978.
[12] Kall, P.: Approximations to Stochastic Programs with Complete Fixed Recourse. Numer. Math. **22** (1974) 333–339.
[13] Kall, P.: Stochastic Linear Programming. Springer-Verlag, 1976.
[14] Kall, P.: Computational Methods for Solving Two-Stage Stochastic Linear Programming Problems. J. Appl. Math. & Phys. **30** (1979) 261–271.
[15] Kleindorfer, G. B.: Bounding Distributions for a Stochastic Acyclic Network. Operations Research **19** (1971) 1586–1601.
[16] Shogan, A. W.: Bounding Distributions for a Stochastic PERT Network. Networks **7** (1977) 359–381.
[17] Stancu-Minasian, I. M.; M. J. Wets: A Research Bibliography in Stochastic Programming. Operations Research **24** (1976) 1078–1119.

[18] VAN SLYKE, R. M.: Monte Carlo Methods and the PERT Problem. Operations Research 11 (1963) 839–860.
[19] VOGEL, W.: Lineares Optimieren. Akademische Verlagsgesellschaft Geest & Portig K. G., 1967.
[20] WETS: Solving Stochastic Programs with Simple Recourse I. Techn. Report, University of Kentucky, Lexington, 1974.
[21] WETS, R.: Solving Stochastic Programs with Simple Recourse II. Proc. John Hopkins Conference Syst. Sc. & Infor., Baltimore, 1975.

Zusammenfassung

Falls für ein Projekt (beschrieben durch eine nicht-leere Menge von Aktivitäten, eine Relation auf dieser Menge von Aktivitäten, deren transitive Hülle eine strikte Ordnung ist, und den einzelnen Aktivitäten zugeordneten Zeitdauern) die Aktivitätszeitdauern als Zufallsvariable vorausgesetzt sind, kann ein zweistufiges stochastisches Programm für ein kostenorientiertes Projektplanungsmodell benutzt werden. Unter Berücksichtigung einer vorgegebenen Zeitschranke für die Zeitdauer des Projektes können für die einzelnen Aktivitätszeitdauern Vorgabezeiten, berechnet werden, so daß die erwarteten Kosten für die Durchführung des Projektes entsprechend der berechneten Vorgabezeiten minimiert werden. Ein Beispiel ist zur Verdeutlichung beigefügt.

Резюме

Если для проекта (описанного непустым множеством работ, отношением на этом множестве работ, транзитивная оболочка которого является строгим порядком, и присоединенными отдельным работам продолжительностями) продолжительности работ предпологаются случайными, применима двухступенчатая программа для ориентированной по стоймостям моделя планирования проекта. В условиях фиксированной границы времени для продолжительности проекта можно для отдельных работ вычислить упреждения, так что ожидаемые стоимости для выполнения проекта по вычисленным упреждениями будут минимальными. Приводится пример для иллюстрации.

Received July 1980, revised November 1981