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Bounding Distributions For Random Project-Completion-Times

Abstract

If the activity-completion-times of a project digraph are random variables the computation of the distribution of the project-completion-time can be difficult. This paper describes possibilities for obtaining lower and upper bounding distributions for the distribution of the project-completion-time. Some approaches, known from the literature, are included as special cases.

Introduction

Graphtheoretical tools have proved to be useful for different areas of applications, see Henn (1968a), (1968b), (1969) for their use in describing and analyzing economic problems, and are adopted here for the coordination and supervision of projects. In this paper a project is given by a set of activities, a binary relation on this set of activities the transitive closure of which is a strict order (irreflexive, asymmetric, transitive) and random activity-completion-times.

The set of activities corresponds with the arc set (at least after the use of dummy arcs) of the following stochastic project digraph

$$D_{1m} = (V, X, f, Y_X)$$

which is a finite, directed, simple, acyclic, weakly connected graph where $V = \{1, \ldots, m\}$ denotes the set of points, X the set of arcs, $f = (f^1, f^2)$, $f^i\colon X \rightarrow V$, $i = 1, 2$, the incidence mapping (with $f^1(x)$, $f^2(x)$ as starting-, end-point of x) of the graph. The points are numbered in such a way that $x \in X \Rightarrow f^1(x) < f^2(x)$. Additionally, $1, m \in V$ represent the single points, called source, sink, with $\{x | f^2(x) = 1, x \in X\} = \emptyset$, $\{x | f^1(x) = m, x \in X\} = \emptyset$.

Different from Henn (1968a) the activities correspond with the arcs and the restrictions between the activities are described by the arc-adjacency relation of the chosen project digraph.

$Y_X = (Y_x | x \in X)$ is a random vector defined on a given probability space (Ω, σ, P) the components of which describe the non-negative activity-completion-times (with degenerate distributions in the non-stochastic case, e.g. for dummies).

If the activity-completion-times are random variables, the project-completion-time (which is yielded by maximizing over the sums of the completion-times of those activities forming source-sink paths) is a random variable the distribution of which is difficult to obtain. Different attempts to describe the situation are known, see e.g. Charnes, Cooper and Thompson (1964) for a first approach using tools from chance-constrained stochastic programming, Cleef and Gaul (1981) for a two-stage stochastic programming model, van Slyke (1963) for one of the first applications of Monte-Carlo methods. Of course, efforts habe been made to determine bounds for the expected project-completion-time, see e.g. Fulkerson (1962) for one of the first approaches besides the well-known PERT-lower bound, Gaul (1981 a), (1981 b) for more recent results, and bounding distributions for the distribution of the project-completion-time, see e.g. Kleindorfer (1971), Shogan (1977). Using appropriate decompositions of the underlying stochastic project digraphs in stochastic subproject digraphs which generalize results obtained in Gaul (1978) this paper describes possibilities for the determination of different bounding distributions including the Kleindorfer- and Shogan-bounding distributions as special cases.

Formulation of the problem

For abbreviation, sometimes only D is written for a digraph in which case V(D) resp. X(D) is used to denote its point resp. arc set. The incidence mapping and the random vector are mostly omitted when, more detailed, (V(D), X(D)) is written instead of D.

For digraphs D^1, D^2 the following notation is used:

$D^1 \subset D^2$ (subdigraph) iff $V(D^1) \subset V(D^2)$, $X(D^1) \subset X(D^2)$,

$D^1 \underset{\cap}{\cup} D^2$ (union-, intersectiondigraph) iff $V(D^1 \underset{\cap}{\cup} D^2) = V(D^1) \underset{\cap}{\cup} V(D^2)$

$$X(D^1 \underset{\cap}{\cup} D^2) = X(D^1) \underset{\cap}{\cup} X(D^2).$$

$D_{ij} \subset D_{1m}$

is called stochastic subproject digraph (s.s.d.) if D_{ij} is a stochastic project digraph with source i, sink j and $V(D_{ij}) \subset \{i, i+1, \ldots, j-1, j\}$,

$$X(D_{ij}) \subset f^{-1}((V(D_{ij}) \times V(D_{ij})) \cap f(X)).$$

The incidence mapping $f/X(D_{ij})$ and the random vector

$$Y_{X(D_{ij})} = (Y_x | x \in X(D_{ij}))$$

are mostly omitted. If for $i, j \in V$ a s.s.d. D_{ij} exists $\tilde{D}_{ij}$ denotes the maximal one (notice $\tilde{D}_{1m} = D_{1m}$).

A path P_{ij} with

$$V(P_{ij}) = \{i_1, \ldots, i_n | i_1 = i, i_n = j\},$$
$$X(P_{ij}) = \{x_1, \ldots, x_{n-1} | f(x_\mu) = (i_\mu, i_{\mu+1}), \mu = 1, \ldots, n-1\}$$

is a special s.s.d.. $(P_{ij})_k$ resp. $(P_{ij})^k$, $k \in V(P_{ij})$, gives the subpath of P_{ij} from i to k resp. k to j. Instead of $D_{f^1(x)f^2(x)}$ the arc notation x is used.

Now, one can define the D_{ij}-completion-time $L(D_{ij}): \mathbb{R}_+^{|X|} \to \mathbb{R}_+$ with

$$(1) \quad L(D_{ij})(Y_X) = \max_{P_{ij} \in D_{ij}} \sum_{x \in X(P_{ij})} \mathrm{proj}_x(Y_X)$$

and its distribution

$$(2) \quad F_{D_{ij}}(t) = P(L(D_{ij}) \leqq t), \qquad t \in \mathbb{R}$$

(with $L(\tilde{D}_{11}) \equiv 0$, $L(\tilde{D}_{1m})$ the project-completion-time).

The computation of $F_{\tilde{D}_{1m}}$ is difficult, but a sequential determination of bounding distributions $F^l_{\tilde{D}_{1\nu}}$, $F^u_{\tilde{D}_{1\nu}}$ for appropriate s.s.d. $\tilde{D}_{1\nu}$, $\nu \in V$, with

$$F^l_{\tilde{D}_{1\nu}}(t) \leqq F_{\tilde{D}_{1\nu}}(t) \leqq F^u_{\tilde{D}_{1\nu}}(t), \qquad t \in \mathbb{R},$$

is possible. Increasing ν up to m yields bounding distributions for the project-completion-time.

Systems of stochastic subproject digraphs

Consider stochastic subproject digraph systems (s.s.d.s.) of the form

$$\delta_\nu = \{D_{i\nu} | D_{i\nu} \text{ is s.s.d., } i < \nu\}, \qquad \nu \in V, \quad \nu > 1.$$

With $B(\delta_v) = \{i | i \in V, D_{iv} \in \delta_v\}$ call δ_v an *appropriate* s.s.d.s. if

(3) $\forall D^1_{i_1v}, D^2_{i_2v} \in \delta_v : D^1_{i_1v} \not\subset D^2_{i_2v}$,

(4) $\forall P_{1v} \subset \tilde{D}_{1m} \; \exists i_0 \in B(\delta_v) \cap V(P_{1v}),\; D_{i_0v} \in \delta_v : P_{1v} = (P_{1v})_{i_0} \cup (P_{1v})^{i_0}$

with $(P_{1v})_{i_0} \cap \delta_v \subset (B(\delta_v), \emptyset),\; (P_{1v})^{i_0} \subset D_{i_0v}$.

Appropriate s.s.d.s. always exist, e.g. $\delta^1_v = \tilde{D}_{1v}$, $\delta^2_v = \{x | x \in X, f^2(x) = v\}$ are appropriate, and generalize *proper* s.s.d.s., described in Gaul (1978), (1981a).

Among the properties for appropriate s.s.d.s. one immediately recognizes

Lemma 1:

If $\delta_v = \{D_{iv}\}$ is an appropriate s.s.d.s. then

(a) $\delta^1_v \supset \delta_v \supset \delta^2_v$,

(b) $j \in V(D_{iv}) \setminus \{i\} \Rightarrow \tilde{D}_{jv} \subset D_{iv}$,

(c) $j \in V(D^1_{i_1v} \cap D^2_{i_2v}) \setminus (\{i_1\} \cap \{i_2\}) \Rightarrow \tilde{D}_{jv} \subset D^1_{i_1v} \cap D^2_{i_2v}$,

(d) $\forall P_{1j} \subset \tilde{D}_{1m} (j \in V(\delta_v)) \; \exists i_0 \in B(\delta_v) \cap V(P_{1j}),\; D_{i_0v} \in \delta_v : P_{1j} = (P_{1j})_{i_0} \cup (P_{1j})^{i_0}$

with $(P_{1j})_{i_0} \cap \delta_v \subset (B(\delta_v), \emptyset),\; (P_{1j})^{i_0} \subset D_{i_0v}$,

(e) $P_{jk} \subset \tilde{D}_{1m},\; x \in X(P_{jk}) \wedge f^2(x) \in V(\delta_v) \setminus B(\delta_v) \Rightarrow \exists D_{i_0v} \in \delta_v$

with $x \in X(D_{i_0v})$,

(f) $P_{jk} \subset \tilde{D}_{1m},\; x \in X(P_{jk}) \wedge f^1(x) \in V(\delta_v) \setminus B(\delta_v) \Rightarrow$ either $\exists D_{i_0v} \in \delta_v$

with $x \in X(D_{i_0v})$ or $(P_{jk})^{f^1(x)} \cap \delta_v = (\{f^1(x)\}, \emptyset)$.

Proof:

(a) Use arguments as described in Gaul (1978), lemma 2.

(b) $j \in V(D_{iv}) \setminus \{i\} \Rightarrow \exists P_{ij} \subset D_{iv}$,

$\tilde{D}_{jv} \not\subset D_{iv} \Rightarrow \exists \mu, \kappa \in V(D_{iv} \cap \tilde{D}_{jv}),\; \mu < \kappa$, and $P_{\mu\kappa} \subset \tilde{D}_{jv}$

with $P_{\mu\kappa} \cap D_{iv} = (\{\mu, \kappa\}, \emptyset)$.

Using the existing paths P_{1i} (with $P_{1i} \cap \delta_v \subset (B(\delta_v), \emptyset)$), $P_{ij} \subset D_{iv}$, $P_{j\mu} \subset D_{iv}$, $P_{\mu\kappa} \subset \tilde{D}_{jv}$, $P_{\kappa v} \subset D_{iv}$ allows to construct

$$\bar{P}_{1\nu} = P_{1i} \cup P_{ij} \cup P_{j\mu} \cup P_{\mu\kappa} \cup P_{\kappa\nu}$$

contradicting (4).

(c) Obvious from (3), (4) and Lemma 1 (b).

(d) Obvious for $j = \nu$ and $j \in B(\delta_\nu)$. For $j \in V(\delta_\nu) \setminus (B(\delta_\nu) \cup \{\nu\})$ choose $P_{j\nu} \subset \tilde{D}_{j\nu}$, see lemma 1 (b), (c), and $P_{1\nu} = P_{1j} \cup P_{j\nu}$. Using (4) one gets

$$P_{1\nu} = (P_{1\nu})_{i_0} \cup (P_{1\nu})^{i_0} \quad \text{and} \quad j \notin V((P_{1\nu})_{i_0}), \quad \text{thus} \quad (P_{1j})_{i_0} = (P_{1\nu})_{i_0}$$

$$\text{with} \quad (P_{1j})_{i_0} \cap \delta_\nu \subset (B(\delta_\nu), \emptyset), \ (P_{1j})^{i_0} = ((P_{1\nu})^{i_0})_j \subset D_{i_0\nu}.$$

(e), (f) Use arguments as described in Gaul (1978), theorem 1. □

A choice from different appropriate s.s.d.s. will be necessary in the following. One gets

Theorem 1:

If $\delta_\nu = \{D_{i\nu}\}$, $\delta'_\nu = \{D'_{j\nu}\}$ are two appropriate s.s.d.s. with $X(\delta_\nu) \supset X(\delta'_\nu)$ then there exists an appropriate s.s.d.s. $\delta''_\nu = \{D''_{q\nu}\}$ with

(a) $X(\delta''_\nu) = X(\delta_\nu)$,

(b) $\forall D'_{j\nu} \in \delta'_\nu \ \exists D''_{q\nu} \in \delta''_\nu : D'_{j\nu} \subset D''_{q\nu}$.

Proof:

If δ_ν satisfies (b) nothing remains to be shown, otherwise there exists $D'_{j_0\nu} \in \delta'_\nu$ with $D'_{j_0\nu} \not\subset D_{i\nu}$ for all $D_{i\nu} \in \delta_\nu$. Choose $D^1_{i_1\nu} \in \delta_\nu$ with $j_0 \in V(D^1_{i_1\nu})$ then $i_1 < j_0 \Rightarrow D'_{j_0\nu} \subset \tilde{D}_{j_0\nu} \subset D^1_{i_1\nu}$ from lemma 1 (b), a contradiction, thus $i_1 = j_0 = i^*$. Consequently, there exist

$$P_{i^*\mu} \subset D'_{i^*\nu}, \ i^* < \mu, \quad \text{with} \quad P_{i^*\mu} \cap D^1_{i^*\nu} = (\{i^*, \mu\}, \emptyset) \quad \text{and therefore}$$

$$D^2_{i_2\nu} \in \delta_\nu \quad \text{with} \quad i^* \in V(D^2_{i_2\nu}) \quad \text{and} \quad D^2_{i_2\nu} \cap D'_{i^*\nu} \supsetneqq (\{i^*, \nu\}, \emptyset),$$

and, again, $i_2 = j_0 = i^*$. Now,

$$\delta^*_\nu = (\delta_\nu \setminus \{D^1_{i^*\nu}, D^2_{i^*\nu}\}) \cup D^*_{i^*\nu} \quad \text{with} \quad D^*_{i^*\nu} = D^1_{i^*\nu} \cup D^2_{i^*\nu}$$

is an appropriate s.s.d.s. satisfying (a) but with a reduced number of s.s.d.. If (b) is not yet satisfied the just described procedure can be repeated. □

The notation $\delta'_\nu \subset\in \delta''_\nu$ is used for appropriate s.s.d.s. δ'_ν, δ''_ν fullfilling the conditions of theorem 1.

Bounding distributions

The usefulness of the s.s.d.s.-approach is based on the following

Theorem 2:

If $\delta_v = \{D_{iv}\}$ is an appropriate s.s.d.s. then

$$L(\tilde{D}_{1v}) = \max_{D_{iv} \in \delta_v} \{L(\tilde{D}_{1i}) + L(D_{iv})\}.$$

Proof:

Use arguments as described in Gaul (1981 a), theorem 1. □

This theorem allows a sequential determination of the $\tilde{D}_{1v}$-completion-time and of lower, upper bounding distributions of $F_{\tilde{D}_{1v}}$ by first determining $\tilde{D}_{1i}$-completion-times and lower, upper bounding distributions of $F_{\tilde{D}_{1i}}$, $i \in B(\delta_v)$, for appropriate s.s.d.s. $\delta_v = \{D_{iv}\}$.

For the stochastic dependence-structure of Y_X association is assumed, see Esary, Proschan and Walkup (1967) which has its main application in reliability theory, see Barlow and Proschan (1978). Additionally, independence for $Y_{X_i} = (Y_x \mid x \in X_i)$, $i = 1, \ldots, k$, for appropriate choosen partitions $X_1, \ldots, X_k$ of X is needed.

Call Y_X, or its set of components $\{Y_x \mid x \in X\}$, associated if

$$\operatorname{cov}(g_1(Y_X), g_2(Y_X)) \geqq 0$$

for all non-decreasing functions $g_1, g_2 : \mathbb{R}^{|X|} \to \mathbb{R}$ for which $E(g_1(Y_X))$, $E(g_2(Y_X))$, $E(g_1(Y_X)\, g_2(Y_X))$ exist.

Needed properties of association are

(A1) Independent random variables are associated.

(A2) Any subset of associated random variables is associated.

(A3) The union of independent sets of associated random variables is associated.

(A4) Non-decreasing functions of associated random variables are associated.

(A5) If Y_X is associated then

$$P(Y_X \leqq y_X) \geqq \prod_{i=1}^{k} P(Y_x \leqq y_x,\ x \in X_i)$$

$$P(Y_X \geqq y_X) \geqq \prod_{i=1}^{k} P(Y_x \geqq y_x,\ x \in X_i)$$

where $X_i \subset X$, $i = 1, \ldots, k$, with $\bigcup_{i=1}^{k} X_i = X$ (and $X_1, \ldots, X_k$ not necessarily a partition).

Now, assume

(5) Y_x, $x \in X$, have finite support

and use, with respect to an appropriate s.s.d.s. $\delta_v = \{D_{iv}\}$, the abbreviations

$$p(y_{X(\delta_v)}) = P(Y_{X(\delta_v)} = y_{X(\delta_v)}),\ y_{X(\delta_v)} = (y_x | x \in X(\delta_v)),$$

$$[\delta_v]_j = \{D_{iv} | D_{iv} \in \delta_v,\ i = j\},$$

then for

(6) $$F^{l}_{\tilde{D}_{1v}}(t) = \sum_{y_{X(\delta_v)}} p(y_{X(\delta_v)}) \max\{0, \sum_{B(\delta_v)} \hat{F}^{l}_{\tilde{D}_{1i}}(t - \max_{[\delta_v]_i}\{L(D_{iv})(y_{X(\delta_v)}, \cdot)\}) - |B(\delta_v)| + 1\},$$

(7) $$F^{l*}_{\tilde{D}_{1v}}(t) = \sum_{y_{X(\delta_v)}} p(y_{X(\delta_v)}) \prod_{B(\delta_v)} \hat{F}^{l}_{\tilde{D}_{1i}}(t - \max_{[\delta_v]_i}\{L(D_{iv})(y_{X(\delta_v)}, \cdot)\}),$$

(8) $$F^{u}_{\tilde{D}_{1v}}(t) = \sum_{y_{X(\delta_v)}} p(y_{X(\delta_v)}) \min_{B(\delta_v)}\{\hat{F}^{u}_{\tilde{D}_{1i}}(t - \max_{[\delta_v]_i}\{L(D_{iv})(y_{X(\delta_v)}, \cdot)\}),$$

where $\hat{F}^{l}_{\tilde{D}_{1i}}$, $\hat{F}^{u}_{\tilde{D}_{1i}}$ are given bounding distributions with

(9) $$\hat{F}^{l}_{\tilde{D}_{1i}}(t) \leqq F_{\tilde{D}_{1i}}(t) \leqq \hat{F}^{u}_{\tilde{D}_{1i}}(t), \quad t \in \mathbb{R},\quad i \in B(\delta_v),$$

one can show

Theorem 3:

If $\delta_v = \{D_{iv}\}$ is an appropriate s.s.d.s., $Y_{X(\delta_v)}$, $Y_{\overline{X(\delta_v)}}$ are independent then

(a) $F^{l}_{\tilde{D}_{1v}}$, $F^{u}_{\tilde{D}_{1v}}$ are bounding distributions with

$$F^{l}_{D_{1v}}(t) \leqq F_{D_{1v}}(t) \leqq F^{u}_{D_{1v}}(t), \quad t \in \mathbb{R},$$

(b) $F^{l*}_{D_{1v}}(t)$ is bounding distribution with

$$F^{l}_{\tilde{D}_{1v}}(t) \leqq F^{l*}_{\tilde{D}_{1v}}(t) \leqq F_{\tilde{D}_{1v}}(t), \quad t \in \mathbb{R},$$

if, additionally, the random vectors $Y_{X(\delta_\nu)}$, $Y_{\overline{X(\delta_\nu)}}$ are associated.

Proof:

Obviously, from the independence assumption

$$(10)\quad F_{\tilde D_{1\nu}}(t) = \sum_{y_{X(\delta_\nu)}} p(y_{X(\delta_\nu)})\, P\Big(\bigcap_{B(\delta_\nu)} \{L(\tilde D_{1i}) \leqq t - \max_{[\delta_\nu]_i} \{L(D_{i\nu})\,(y_{X(\delta_\nu)}, \cdot)\}\}\Big)$$

and from the well-known Fréchet-bounds and (9)

$$F_{\tilde D_{1\nu}}(t) \leqq \sum_{y_{X(\delta_\nu)}} p(y_{X(\delta_\nu)}) \min_{B(\delta_\nu)} \{F_{\tilde D_{1i}}(t - \max_{[\delta_\nu]_i} \{L(D_{i\nu})\,(y_{X(\delta_\nu)}, \cdot)\})\} \leqq F^{u}_{\tilde D_{1\nu}}(t)$$

$$F_{\tilde D_{1\nu}}(t) \geqq \sum_{y_{X(\delta_\nu)}} p(y_{X(\delta_\nu)}) \max\{0, \sum_{B(\delta_\nu)} F_{\tilde D_{1i}}(t - \max_{[\delta_\nu]_i} \{L(D_{i\nu})\,(y_{X(\delta_\nu)}, \cdot)\})$$

$$- |B(\delta_\nu)| + 1\} \geqq F^{l}_{\tilde D_{1\nu}}(t).$$

Moreover, as $\hat F^{l}_{\tilde D_{1i}}$, $\hat F^{u}_{\tilde D_{1i}}$ are distributions and the properties of distributions are maintained under the summation- and minimum-operation with respect to $i \in B(\delta_\nu)$, $F^{l}_{\tilde D_{1\nu}}$ and $F^{u}_{\tilde D_{1\nu}}$ are distributions. If, additionally, $Y_{X(\delta_\nu)}$, $Y_{\overline{X(\delta_\nu)}}$ are associated one concludes:

Y_X is associated because of (A3).

As $L(D_{ij})$ is non-decreasing for arbitrary s.s.d. D_{ij} one gets

$$F_{\tilde D_{1\nu}}(t) \geqq \sum_{y_{X(\delta_\nu)}} p(y_{X(\delta_\nu)}) \prod_{B(\delta_\nu)} F_{\tilde D_{1i}}(t - \max_{[\delta_\nu]_i} \{L(D_{i\nu})\,(y_{X(\delta_\nu)}, \cdot)\}) \geqq F^{l*}_{\tilde D_{1\nu}}(t)$$

because of (A2), (A4), (A5) and (9), (10). For the same reasons as mentioned above $F^{l*}_{\tilde D_{1\nu}}$ describes a distribution function.

To show that $F^{l*}_{\tilde D_{1\nu}}$ improves $F^{l}_{\tilde D_{1\nu}}$ notice that

$$(11)\quad \alpha_i \in [0,1],\quad i = 1, \ldots, r \Rightarrow \prod_{i=1}^{r} \alpha_i \geqq \sum_{i=1}^{r} \alpha_i - r + 1$$

(An easy induction argument shows that if (11) is valid for $r = n$ it follows for $r = n+1$

$$\Big(1 - \prod_{i=1}^{n} \alpha_i\Big)(1 - \alpha_{n+1}) \geqq 0 \Rightarrow \prod_{i=1}^{n+1} \alpha_i \geqq \prod_{i=1}^{n} \alpha_i + \alpha_{n+1} - 1$$

$$\geqq \sum_{i=1}^{n} \alpha_i - n + 1 + \alpha_{n+1} - 1$$

$$= \sum_{i=1}^{n+1} \alpha_i - (n+1) + 1.)$$

from which one gets

$$F^{1*}_{\tilde{D}_{1\nu}}(t) = \sum_{y_{X(\delta_\nu)}} p(y_{X(\delta_\nu)}) \prod_{B(\delta_\nu)} \hat{F}^{1}_{\tilde{D}_{1i}}(t - \max_{[\delta_\nu]_i} \{L(D_{i\nu})(y_{X(\delta_\nu)}, \cdot)\})$$

$$\geqq \sum_{y_{X(\delta_\nu)}} p(y_{X(\delta_\nu)}) \max\{0, \sum_{B(\delta_\nu)} \hat{F}^{1}_{\tilde{D}_{1i}}(t - \max_{[\delta_\nu]_i} \{L(D_{i\nu})(y_{X(\delta_\nu)}, \cdot)\})$$

$$- |B(\delta_\nu)| + 1\} = F^{1}_{\tilde{D}_{1\nu}}(t). \qquad \square$$

Notice, that for theorem 3 the knowledge of bounding distributions $\hat{F}^{1}_{\tilde{D}_{1i}}$, $\hat{F}^{u}_{\tilde{D}_{1i}}$, $i \in B(\delta_\nu)$, is presupposed. Here, a sequential determination of the bounding distributions is possible if bounding distributions

$$\hat{F}^{1}_{\tilde{D}_{1j(i)}}, \hat{F}^{u}_{\tilde{D}_{1j(i)}}, j(i) \in B(\delta_i), \text{ for appropriate s.s.d.s. } \delta_i, \ i \in B(\delta_\nu),$$

are known or can be computed and the needed stochastic dependence assumptions are fulfilled.

Sufficient conditions, known from the literature, are

independence for $Y_{X(\delta_2^2)}, \ldots, Y_{X(\delta_m^2)}$ and

association for the single random vectors $Y_{X(\delta_\nu^2)}$, $\nu \in \{2, \ldots, m\}$,

together with the starting distributions

$$\hat{F}^{1}_{\tilde{D}_{11}}(t) = \hat{F}^{u}_{\tilde{D}_{11}}(t) = \begin{cases} 1, & t \geqq 0 \\ 0, & \text{otherwise.} \end{cases}$$

For the special case $\delta_\nu = \delta_\nu^2$ (6), (7), (8) describe the bounding distributions of Shogan (1977). The bounding distributions of Kleindorfer (1971) are not explicitly presented because, although they were developped under the stronger assumtion of independence for Y_x, $x \in X$, see also (A1), the bounding distributions (7), (8) are tighter.

Of course, the question arises how the bounding distributions are influenced by the special choice of an appropriate s.s.d.s..

In the special case when $\delta'_\nu = \{D'_{j\nu}\}$, $\delta''_\nu = \{D''_{q\nu}\}$ are proper s.s.d.s. with $\delta'_\nu \subset\in \delta''_\nu$ and X^* is choosen in such a way that

(12) $X(\delta'_\nu) \cup X^* = X(\delta''_\nu)$, $X(\delta'_\nu) \cap X^* = \emptyset$,

using the abbreviations

$F'^{u}_{\tilde{D}_{1v}}$, $F''^{u}_{\tilde{D}_{1v}}$, $F'^{1*}_{\tilde{D}_{1v}}$, $F''^{1*}_{\tilde{D}_{1v}}$ to indicate the underlying s.s.d.s.,

$p(y_{X^*}) = P(Y_{X^*} = y_{X^*})$,

$p(y_{qj}) = P(Y_{X(\tilde{D}_{qj})} = y_{qj})$, $y_{qj} = (y_x \mid x \in X(\tilde{D}_{qj}))$

for $q \in P(D''_{qv})$, $D''_{qv} \in \delta''_v$, $j \in P(D''_{qv}) \cap B(\delta'_v)$

one can show

Theorem 4:
If $\delta'_v = \{D'_{jv}\}$, $\delta''_v = \{D''_{qv}\}$ are proper s.s.d.s. with $\delta'_v \subset\subset \delta''_v$, $Y_{X(\delta'_v)}$, Y_{X^*}, $Y_{\overline{X(\delta''_v)}}$ are independent where X* is choosen according to (12) then

(a) $F''^{u}_{\tilde{D}_{1v}}(t) \leq F'^{u}_{\tilde{D}_{1v}}(t)$, $t \in \mathbb{R}$,

if additionally

$$\hat{F}^{u}_{\tilde{D}_{1j}}(t) \geqq \sum_{y_{qj}} p(y_{qj})\, \hat{F}^{u}_{\tilde{D}_{1q}}(t - L(\tilde{D}_{qj})\,(y_{qj}, \cdot))$$

for $q \in P(D''_{qv})$, $D''_{qv} \in \delta''_v$, $j \in P(D''_{qv}) \cap B(\delta'_v)$,

(b) $F''^{1*}_{\tilde{D}_{1v}}(t) \geqq F'^{1*}_{\tilde{D}_{1v}}(t)$, $t \in \mathbb{R}$,

if additionally

$$\hat{F}^{1}_{\tilde{D}_{1j}}(t) \leqq \sum_{y_{qj}} p(y_{qj})\, \hat{F}^{1}_{\tilde{D}_{1q}}(t - L(\tilde{D}_{qj})\,(y_{qj}, \cdot))$$

for $q \in P(D''_{qv})$, $D''_{qv} \in \delta''_v$, $j \in P(D''_{qv}) \cap B(\delta'_v)$,

and the random vectors $Y_{X(\delta'_v)}$, Y_{X^*}, $Y_{\overline{X(\delta''_v)}}$ are associated.

The rather technical proof is omitted.

The question whether a similar statement is true for appropriate s.s.d.s. is under consideration.

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