OPTIMAL ROUTES IN COMPOUND TRANSPORTATION SYSTEMS

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Abstract

Usually, in realistic transport situations different transportation systems are available which form a compound transportation system.

Thus, the problem of finding an optimal route from one node of the compound transportation system to another involves not only costs (time) of using the actually chosen transportation system but also costs (time) of changing between different transportation systems.

An efficient solution procedure for an optimal (cheapest, shortest) route in a compound transportation system - which, now, can contain cycles - is presented and demonstrated by means of an example.

1. Introduction

Starting point of the following considerations is the possibility of describing multicommodity flow problems by 'huge' linear programs where 'huge' refers to the number of columns of the restriction matrix of the problem. These columns correspond to the possible routes between pairs of sources and sinks of different flows (see e.g. Ford and Fulkerson [10] or Gaul [12]) the determination of which would cause an enormous effort prior to the actual solution of the problem. Thus, e.g. for interchanging these columns within the rules of the revised simplex method a special procedure for obtaining the momentarily desired source-sink multicommodity flow routes is needed.

Of course, algorithms for optimal routes in networks enjoy an everlasting interest (see e.g. Dantzig [4], Dijkstra [6] as early references, Dreyfuss [9] for one of the first excellent appraisals of shortest route algorithms, Dial, Glover, Karney and Klingman [5] for a

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paper emphasizing the computer implementation viewpoint, Domschke [7], [8] for applications in the logistics area, Gaul [11] for one of the attempts to consider additional constraints within route problems).

In this paper it is assumed that in realistic transport situations different transportation systems are available which form a compound transportation system and - on the basis of these assumptions - an optimal route algorithm is developed which pays attention to the possibilities of changing between different transportation systems. Now, not only costs (time) of using the actually chosen transportation system but also costs (time) of changing from one transportation system to another must be considered. Additional sources of costs (time) components can be incorporated, as mentioned within the multi-commodity flow problem framework described in the next section, allowing different degrees of complexity for handling such transportation situations (see e.g. Beckmann [1], [2], [3] for the discussion of lots of problems in the field of traffic flow).

Having formulated the compound transportation systems approach in section 2 in section 3 an optimal route algorithm is presented to overcome the exchange problem with respect to the possible use of different transportation systems. In section 4 an example of simplest form is handled for illustration.

2. Compound Transportation Systems

If the structure of the i-th special transportation system is described by a finite simple directed graph $G^i = (N^i, A^i, J^i)$ where N^i denotes the set of nodes, A^i the set of arcs and $J^i: A^i \to N^i \times N^i \times \{i\}$ $(a^i \mapsto (J_1^i(a^i), J_2^i(a^i); i)$ with $J_1^i(a^i)$ and $J_2^i(a^i)$ as start and end nodes of arc a^i) the incidence mapping of G^i, $i=1,\dots,I$, then the finite directed graph $G = (N,A,J)$ with node set $N = \bigcup_{i=1}^{I} N^i$, arc set $A = \bigcup_{i=1}^{I} A^i$ and incidence mapping $J: A \to N \times N \times I$ with $J_j(a) = J_j^i(a)$, $a \in A^i$, $j=1,2$, represents a compound transportation system (see Fig.2(e) for an example of a graph composed by the graphs of Fig.2(a),(b),(c),(d)). Generally, G is no longer simple, i.e. parallel arcs will exist which, however, can be distinguished by the already introduced notation $(n_1, n_2; i)$ for an arc $a^i \in A^i$ from node n_1 to node n_2.

Now, a route from node n_1 to node n_{k+1} can be described by the sequence of arcs a^{i_1} $(= (n_1,n_2;i_1))$, $a^{i_2}(= (n_2,n_3;i_2)),\ldots,a^{i_k}(= (n_k,n_{k+1};i_k))$ traversed in the given order. If n_1 is equal to n_{k+1} the route is called a cycle. If each node (arc) of the route is used only once the route is called cycle-free (simple). Standard optimal route algorithms search for cycle-free optimal routes but in this context the larger set of all simple routes has to be taken into consideration (see Fig.1 where for each route from node 1 to node 3 one has to change transportation systems but changing in node 2 (a main junction) can be less convenient than changing in node 5 (a state-subsidized fringe area). see also Fig. 3 (a) and the remarks given there).

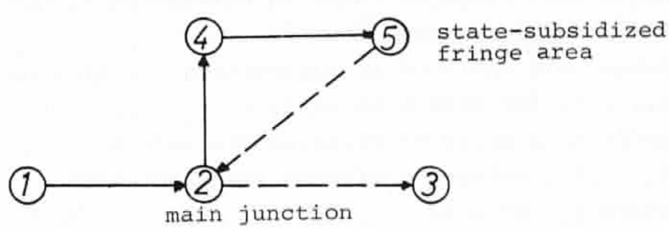


Fig. 1: Transportation System 1 (--->), Transportation System 2 (——>)

A multi-commodity flow problem in a compound transportation system can be described as follows:
For $m = 1,\ldots,M$ let $n_m,\hat{n}_m \in N$, $n_m \neq \hat{n}_m$, be the source and sink of flow of commodity m - flow m (for abbreviation)-.

Let $r_{m,l}$ be the l-th route from n_m to $\hat{n}_m$ and $A(r_{m,l})$ its arc set, $l=1,\ldots,l(m)$. If $f_{m,l}$ is the flow value of flow m along route $r_{m,l}$ then

$$f = \begin{pmatrix} \vdots \\ f_m \\ \vdots \end{pmatrix} \quad \text{with} \quad f_m = \begin{pmatrix} \vdots \\ f_{m,l} \\ \vdots \end{pmatrix}$$

is called M-commodity flow (in route representation).

Of course, additional restrictions, costs viewpoints, etc., will be of interest, e.g.:

If

$c \;\; : A \to \mathbb{R}_+$ denotes capacity on arcs,

$cc \;\; : B \to \mathbb{R}_+$ denotes capacity of changing for suited arc pairs of $B = \{(a^{i}1, a^{i}2) | J_2(a^{i}1) = J_1(a^{i}2)$, changing from $G^{i}1$ to $G^{i}2$ is possible$\}$,

$d \;\; : \{1,\dots,H\} \to \mathbb{R}_+$ denotes stockpile of resources for transportation on arcs,

$dd \;\; : \{1,\dots,HH\} \to \mathbb{R}_+$ denotes stockpile of resources for changing between different transportation systems,

if

$z^m \;\; : A \to \mathbb{R}_+$ denotes unit costs of transportation for flow m on arcs

$zz^m : B \to \mathbb{R}_+$ denotes unit costs of changing between different transportation systems for flow m,

$z^{mh} : A \to \mathbb{R}_+$ denotes unit costs of transportation-resource h, h=1,...,H, for flow m on arcs,

$zz^{mh}: B \to \mathbb{R}_+$ denotes unit costs of changing-resource h, h=1,...,HH, between different transportation systems for flow m,

then, with the matrices

$$W^m = (w_{a,r_{m,l}}) \quad \text{with} \quad w_{a,r_{m,l}} = \begin{cases} 1, & a \in A(r_{m,l}) \\ 0, & \text{otherwise} \end{cases}$$

(rows ≙ arcs, columns ≙ routes),

$$WW^m = (ww_{a^{i}1a^{i}2,r_{ml}}) \quad \text{with} \quad ww_{a^{i}1a^{i}2,r_{m,l}} = \begin{cases} 1, & a^{i}1, a^{i}2 \in A(r_{m,l}) \\ 0, & \text{otherwise} \end{cases}$$

(rows ≙ arc pairs of B, columns ≙ routes),

$R^m = (r^m_{h,a})$ with $r^m_{h,a}$ as consumption of resource h for transportation of flow m on arc a

(rows ≙ resources, columns ≙ arcs)

$RR^m = (rr^m_{h,a^{i}1a^{i}2})$ with $rr^m_{h,a^{i}1a^{i}2}$ as consumption of resource h for changing between different transportation systems

(rows ≙ resources, columns ≙ arc pairs of B)

one can formulate restrictions of the kind

$$\sum_{m=1}^{M} W^m f_m \leq c \tag{1}$$

$$\sum_{m=1}^{M} WW^m f_m \leq cc \tag{2}$$

$$\sum_{m=1}^{M} R^m W^m f_m \leq d \tag{3}$$

$$\sum_{m=1}^{M} RR^m WW^m f_m \leq dd \tag{4}$$

and it is easy to check that

$$z(r_{m,1}) = \sum_{a \in A} w_{a,r_{m,1}} [z_a^m + \sum_{h=1}^{H} r_{h,a}^m z_a^{mh}] \tag{5}$$

$$+ \sum_{(a^i{}_1, a^i{}_2) \in B} ww_{a^i{}_1 a^i{}_2, r_{m,1}} [zz^m_{a^i{}_1 a^i{}_2} + \sum_{h=1}^{HH} rr^m_{h,a^i{}_1 a^i{}_2} zz^{mh}_{a^i{}_1 a^i{}_2}]$$

gives the unit costs of transportation for flow m along route $r_{m,1}$.

If

$g: \{1,\dots,M\} \to \mathbb{R}_+$ denotes minimum requirements for the M-commodity flow at sinks $\hat{n}_m$, $m=1,\dots,M$,

restrictions of the kind

$$(1,\dots,1) f_m \geq g_m \quad , \quad m = 1,\dots,M, \tag{6}$$

can be added which do not alter the costs expression (5).

$$\min\{ \sum_{m=1}^{M} (\dots, z(r_{m,1}), \dots) f_m \mid \text{Restrictions of the kind (1),(2),(3),(4),(6) have to be satisfied.}\} \tag{7}$$

describes one of the possible linear programming approaches for such a multicommodity flow in a compound transportation system situation.

The explicit presentation of such restrictions, costs viewpoints, etc., is just to show how such conditions can be handled by corresponding linear programs. However, for recording a linear program of the form as given in (7) all simple routes $r_{m,l}$ must be determined in advance, an enormous effort prior to the actual solution of the problem.

With simplex multipliers $\pi \in \mathbb{R}^{|A|}$, $\lambda \in \mathbb{R}^{|B|}$, $\mu \in \mathbb{R}^{H}$, $\nu \in \mathbb{R}^{HH}$, $\sigma \in \mathbb{R}^{M}$ assigned to the restrictions (1), (2), (3), (4), (6)

$$z(r_{m,l}) - \sum_{a \in A} w_{a,r_{m,l}} [\pi_a + \sum_{h=1}^{H} r^m_{h,a} \mu_h] - \sum_{(a^{i}1,a^{i}2) \in B} ww_{a^{i}1a^{i}2,r_{m,l}} [\lambda_{a^{i}1a^{i}2} + \sum_{h=1}^{HH} rr^m_{h,a^{i}1a^{i}2} \nu_{a^{i}1a^{i}2}] - \sigma_m < 0 \tag{8}$$

should be valid for a route $r_{m,l}$ to be a candidate for an interchange of columns within the rules of the revised simplex method applied to problem (7). From (5) and (8) it follows that values for arc pairs

$$v(a^{i}1,a^{i}2) = z^m_{a^{i}1} - \pi_{a^{i}1} + \sum_{h=1}^{H} r^m_{h,a^{i}1} [z^{mh}_{a^{i}1} - \mu_h] + zz^m_{a^{i}1a^{i}2} - \lambda_{a^{i}1a^{i}2} + \sum_{h=1}^{HH} rr^m_{h,a^{i}1a^{i}2} [zz^{mh}_{a^{i}1a^{i}2} - \nu_{a^{i}1a^{i}2}] \tag{9}$$

can be defined and an optimal route procedure which labels arcs (instead of nodes as performed by standard route algorithms) can be developed as described in the next section. If

$$\min_{l=1,\ldots,l(m)} \{ \sum_{(a^{i}1,a^{i}2) \in A(r_{m,l})} v(a^{i}1,a^{i}2) \} \geq \sigma_m \quad , m = 1,\ldots,M$$

optimality in problem (7) is reached. Otherwise M optimal routes in the compound transportation system have to be determined.

For simplicity, the use of resource restrictions is omitted in the following but it should now be clear how additional restrictions can be included in the problem formulation.

3. Optimal Route Procedure

For the problem of finding an optimal route from node n_1 to node n_2 in a compound transportation system a restriction is made to the main aspects 'transportation on arcs' and 'changing between different transportation systems' (see (1), (2) and (5) when summation with respect to h is omitted). Also, the further description does not depend on the choice of a special flow m, thus, z_a ($zz_{a^{i_1}a^{i_2}}$) denotes unit costs of transportation on arc a (unit costs of changing from arc a^i1 to arc a^i2). For computational convenience two additional arcs

$$a^{START} \text{ with } J_2(a^{START}) = n_1, \quad a^{STOP} \text{ with } J_1(a^{STOP}) = n_2,$$

are introduced where $z_{a^{START}}$ ($z_{a^{STOP}}$) can be interpreted as warehousing expenses in node n_1 (node n_2) and $z_{a^{START}a}$ with $J_1(a) = n_1$ ($z_{aa^{STOP}}$ with $J_2(a) = n_2$) as loading (unloading) charges. Notice, that the set B (see the explanations of additional restrictions subsequent to the definition of an M-commodity flow) has to be augmented analogously.

Procedure:

STEP 0:

$L = \{a^{START}\}$

$l(a^{START}) = (V_{a^{START}}, \emptyset) \quad \text{with } V_{a^{START}} = z_{a^{START}}$

$\bar{A} = \{a \mid a \in A, J_1(a) = J_2(a^{START})\}$

$s = 1$

STEP s:

$\bar{A} = \emptyset \rightarrow$ STOP (NO SOLUTION)

$$\min_{(a^i1, a^i2) \in L \times \bar{A} \cap B} \{V_{a^{i_1}} + zz_{a^{i_1}a^{i_2}} + z_{a^{i_2}}\}$$

$$= V_{a^{i_1*}} + zz_{a^{i_1*}a^{i_2*}} + z_{a^{i_2*}} = V_{a^{i_2*}}$$

(As the minimum is not necessarily unique, $P(a^i2^*)$ is used to denote the set of predecessor arcs of a^i2^* with $a^i1^* \in P(a^i2^*)$.)

$L = L \cup \{a^i2^*\}$

$l(a^i2^*) = (V_{a^{i_2*}}, P(a^i2^*))$

$a^{i_2^*} = a^{STOP} \rightarrow STOP$ (OPTIMAL SOLUTION)

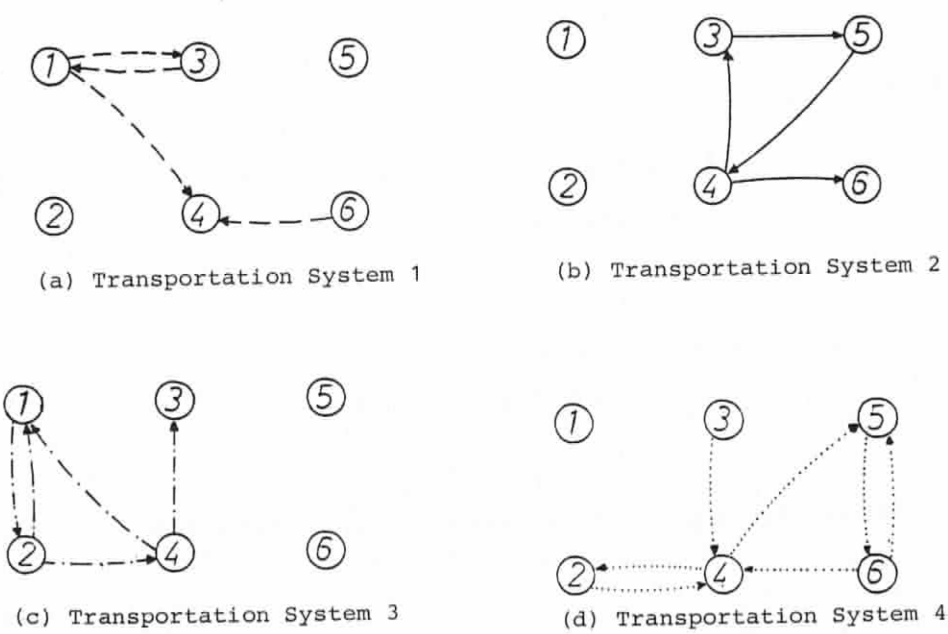
$\mathbb{A} = ((\mathbb{A}\setminus\{a^{i_2^*}\}) \cup \{a \mid a \in A, J_1(a) = J_2(a^{i_2^*})\})\setminus L$

$s = s + 1 \rightarrow$ GO TO STEP s.

Here, L denotes the set of labeled arcs, the vector l(a) contains the costs V_a of an optimal route from a^{START} to a and the set of predecessor arcs P(a). The procedure stops if either n_1 and n_2 are not connected or the final arc a^{STOP} is reached. If a^{STOP} is labeled backtracking along the predecessor arcs establishes an optimal route.

4. Example

The compound transportation system of Fig. 2(e) is composed by four different transportation systems which are described by the finite simple directed graphs of Fig. 2(a),(b),(c),(d).



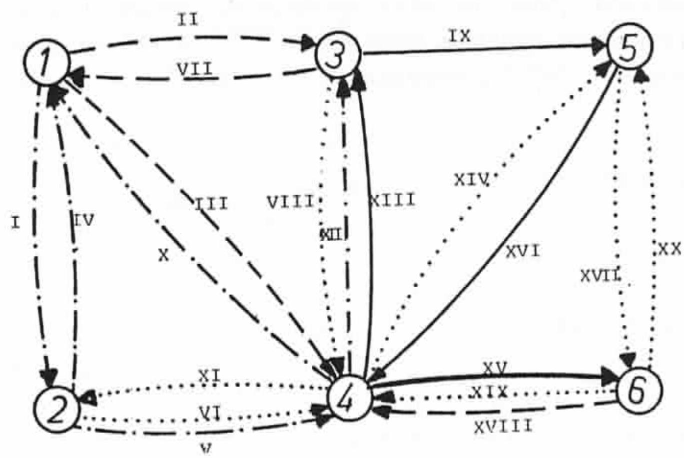
(a) Transportation System 1

(b) Transportation System 2

(c) Transportation System 3

(d) Transportation System 4

Fig. 2: Compound Transportation System



(e)

Fig. 2: Compound Transportation System

Roman numerals are assigned to the arcs of the compound transportation system which help to identify the unit costs of changing from the arcs in the left-hand column to the arcs in the top row in Tab. 1.

	$_a$START	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV	XV	XVI	XVII	XVIII	XIX	XX	$_a$STOP
$_a$START	1	1	1	2																		
I		2			1	1	3															
II			7					1	3	4												
III				14							8	6	1	4	6	4						
IV		1	4	4	5																	
V						1					1	5	4	1	4	13						
VI							3				6	1	5	2	1	13						
VII		3	1	1				7														
VIII									3		3	3	4	1	3	3						
IX										2							1	4				
X		1	6	5							8											
XI					3	2	1					3										
XII								1	3	3			5									
XIII								4	4	3				2								
XIV															5		3	2				
XV																4			1	5	4	1
XVI											3	3	2	1	3	2	1					
XVII																		6	5	1	5	2
XVIII											6	5	1	2	5	1			4			
XIX											3	1	5	2	1	6				6		
XX																	2	1			6	
$_a$STOP																						2

Tab. 1: Unit costs of Changing between Different Transportation Systems, Unit costs of Transportation (in the Diagonal)

Tab. 1 already indicates that in this example an optimal route from node 1 to node 6 is wanted because from arc a^{START} a changing to arcs I, II, III, from arcs XV, XVII a changing to arc a^{STOP} is possible.

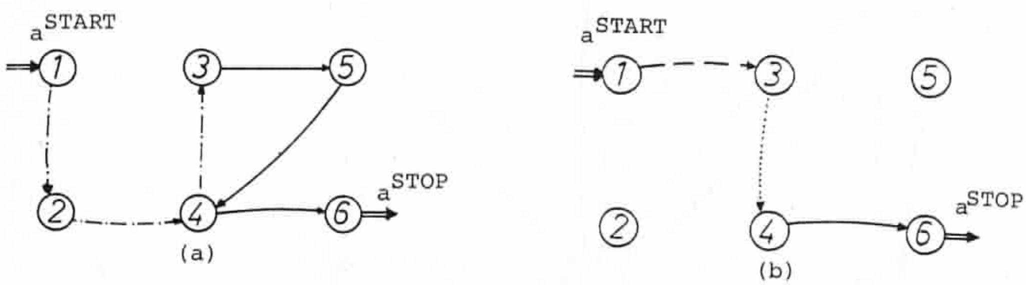


Fig. 3: Optimal Routes from Node 1 to Node 6

The optimal solution is shown in Fig. 3. There are only two optimal routes from node 1 to node 6 (from arc a^{START} to arc a^{STOP}, respectively) with optimal costs of 25 units of account.

In the optimal route of Fig. 3(a) node 4 is traversed twice but advantageously only one changing from transportation system 3 to transportation system 2 is necessary. The optimal route of Fig. 3(b) is cycle-free, but twice changing the transportation systems (from system 1 to system 4 to system 2) is needed. The discussion whether cycles or further changings (possibilities of damaging, etc.) are preferable is left to the reader.

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