FINANCIAL PLANNING VIA STOCHASTIC PROGRAMMING: A STOCHASTIC FLOWS-WITH-GAINS APPROACH

W. Gaul
Institut für Entscheidungstheorie und
Unternehmensforschung, Universität Karlsruhe
D-7500 Karlsruhe

Abstract

A stochastic version of a financial planning problem is explicitly handled by a stochastic flows-with-gains approach because it is hoped that the use of network flow formulations will increase the readiness of adoption of mathematical programming procedures in this area.

The possibility of using flows-with-gains for the formulation of financial decisions is known but its combination with stochastic programming, which allows a more realistic description of the stochastic aspects of the underlying problem, is new.

An example is included for illustration.

Graphtheoretical Framework for Financial Decisions

Many of the financial decision problems can be formulated as variants of linear programs (at least after proper transformations) and most of them can also be presented by the help of graphtheoretical tools because of the resemblance between financial transactions and network flows (see e.g. CHRISTOFIDES/HEWINS/SALKIN (1979), CRUM/KLINGMAN/TAVIS (1979), DENK (1978), GAUL (1983), GOLDEN/LIBERATORE/LIEBERMAN (1979), HORST (1975 a), SRINIVASAN (1974)). Authors using the graphtheoretical framework argue that - on the one hand - the ability to draw a visual representation of the problem facilitates communication and understanding and - on the other hand - graphtheoretical methods are superior in computational speed to state-of-the art linear programming codes.

Whereas in some cases the possibility of modeling problems within the area of banking and finance by graphtheoretical or networktheoretical formulations is only mentioned (see e.g. GREGORY (1976) for cash flow models, COHEN/MAIER/VAN DER WEIDE (1981) for bank operations and consulting services) in CRUM/KLINGMAN/TAVIS (1979) an impressive pleading for network descriptions (wheresoever possible) of real world applications, expecially for large-scale financial planning models, and some experience with the acceptance by managers is given.

Lecture Notes in Economics and Mathematical Systems, Vol. 227
Risk and Capital. Edited by G. Bamberg and K. Spremann.

Although already JEWELL (1962) (who developed the first algorithm for the solution of the flows-with-gains or generalized network flow problem) mentions financial budgeting in a warehouse operation as an example for the application of his flows-with-gains approach and RUTENBERG (1970) uses a networktheoretical description for his maneuvering of liquid assets in a multi-national company SRINIVASAN (1974) (see also the re-writing of the SRINIVASAN-paper by DENK (1978)) seems to be the first who explicitly solved his example for cash management decisions using a transshipment code. GOLDEN/LIBERATORE/LIEBERMAN (1979) claim that compounding of interest and reinvestment of returns is not included in the SRINIVASAN-formulation and use a version of the flows-with-gains approach for a better modeling of the cash flow management situation. A flows-with-gains algorithm (and other networktheoretical algorithms) is also used in HORST (1975 a) and in CHRISTOFIDES/HEWINS/SALKIN (1979) for different types of deterministic arbitrage (trading in currencies in order to obtain profit from discrepancies in different markets (space arbitrage), discrepancies because of different maturities (time arbitrage) and discrepancies between yields on short term investment in different currencies (interest arbitrage)).

All up to now mentioned approaches are non-stochastic except for GAUL (1983) where - for the first time - a combination of the short term financial planning transshipment approach and the short term financial planning under uncertainty approach is described using the stochastic flow algorithm of CLEEF/GAUL (1980) applied to a modified SRINIVASAN-example.

In this paper a stochastic flows-with-gains approach is used to take into consideration the uncertainty element related to most financial planning decisions and the already mentioned possibilities of including compounding of interest and reinvestment of returns in the problem formulation. A modification of the network transformation of the horizon model for capital budgeting of WEINGARTNER (1963) used in CRUM/KLINGMAN/TAVIS (1979) is combined with the cash flow management example of GOLDEN/LIBERATORE/LIEBERMAN (1979) and serves for illustration.

Financial Planning via Stochastic Programming

Having mentioned the possibility of modeling financial decision problems by the help of linear programming formulations it has to be emphasized that stochastic linear programming should allow a more realistic representation of the degree of uncertainty adhered to the prediction of cash flows, the forecasts of interest rates, and so on. Of course, non-stochastic descriptions of financial planning situations are easier to handle, of course, for short term planning the uncertainty element is less important (see e.g. ORGLER (1970), SRINIVASAN (1974) for arguments in that direction), but more and more researchers include stochastic aspects in their model presentations (see e.g. ZIEMBA/VICKSON (Eds.) (1975), LEVY/SARNAT (Eds.) (1977)) or, at least, mention possibilities for additional stochastic considerations (see e.g. SARTORIS/HILL (1983) for a recent paper).

Especially, stochastic linear programming formulations have obtained great interest (see e.g. STANCU-MINASIAN/WETS (1976) where about 800 references are given (an up-to-date bibliography is under preparation)). The terms - distribution model - two stage model with recourse - chance constrained model - distinguish between the main areas of stochastic programming (see e.g. KALL (1976)) which all have been used for the description of financial planning problems (recent examples are e.g. KALLBERG/WHITE/ZIEMBA (1982) for a two stage model with recourse approach, BRICK/MELLON/SURKIS/MOHL (1983) for a chance constrained model approach, ALBACH (1967) is an early, BÜHLER (1981) a more recent German contribution).

In GAUL (1983) the short term financial planning under uncertainty problem was handled by the two stage model with recourse approach within a graphtheoretical framework using a normal flow formulation (without gains) for description.

In the following a stochastic flows-with-gains approach will be presented which combines the two stage model with recourse approach of stochastic programming with a flows-with-gains approach for the solution of a typical financial planning situation.

Stochastic Flows-with-Gains Approach

The first notice of the fact that the flows-with-gains or generalized network flow approach could be used for the formulation and solution of financial planning problems was already given by JEWELL (1962) who developed the first flows-with-gains algorithm (see e.g. [12], [14], [15], [17], [21], [22], [23], [24], [30] for further engagement in the flows-with-gains topic), the explicit application to financial decision situations in the papers [4], [7], [10], [13], [26] has already been mentioned.

If - for the visual representation - nodes are used to describe sources of financing (e.g. cash assets, receipts of payments from customers,...) and uses of funds (e.g. payments of wages, bills for raw material, taxation,...) as well as interesting time points of planning periods (e.g. maturity dates, ...), and, arrows are used to describe financial transactions (e.g. cash flows, borrowing and lending activities, transformations between assets of different liquidity levels and maturity dates,...) between the just mentioned nodes, many of the characteristic features of financial planning situations can be figured by the help of network-flow-theoretical tools.

In this context it is remarkable that, generally, a financial transaction f_{ij} leaving node i (out-flow from i) undergoes a gain g_{ij} (or loss) before entering node j (in-flow to j) because additional transactions (payments of borrowing or lending rates, costs for liquidity and/or maturity transformations, ...) have to be taken into consideration. The in-flow to j is $g_{ij}f_{ij}$.

The situation is pictured in the following figure.

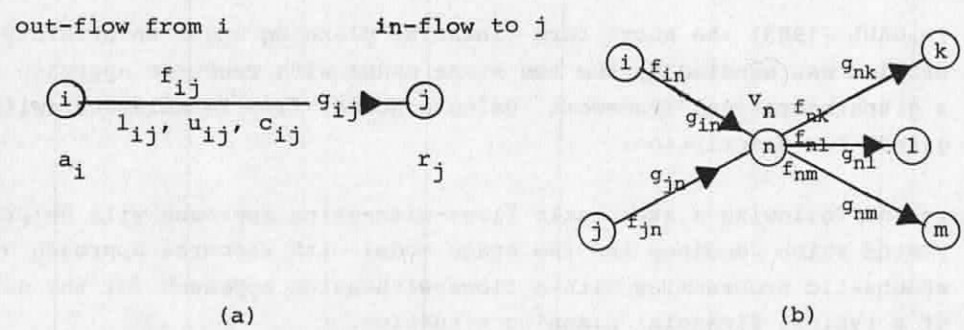

FIG.1: Data of the Problem.

In FIG. 1(a) a_i denotes the amount of financial availabilities of node i, r_j the amount of financial requirements of node j, l_{ij}, u_{ij} and c_{ij} denote a lower, upper bound and costs for the financial transaction f_{ij} (out-flow from i) with gain g_{ij}. If

$$\left.\begin{array}{r} g_{ij} > 1 \\ g_{ij} = 1 \\ 0 < g_{ij} < 1 \\ g_{ij} = 0 \\ g_{ij} < 0 \end{array}\right\}, \text{ the financial transaction } f_{ij} \text{ (out-flow from i) } \left\{\begin{array}{l} \text{undergoes a gain,} \\ \text{remains unchanged,} \\ \text{undergoes a loss,} \\ \text{can be interpreted as slack-variable,} \\ \text{creates a reversal for the in-flow } g_{ij}f_{ij} \text{ to j.} \end{array}\right. \quad (1)$$

To employ a more formal notation a directed graph $G = (N,A,I)$ (with node set N, arrow set A and incidence mapping $I = (I^1,I^2)$ with mappings $I^1,I^2: A \to N$ which indicate the starting node $I^1(a)$ and the end node $I^2(a)$ for every arrow $a \in A$ of graph G) is used for the description. In the non-stochastic case the flows-with-gains problem corresponds to the following mathematical programming formulation

$$\sum_{a \in A} c_a f_a = \min! \quad (2)$$

$$\sum_{\{a|a\in A, I^1(a)=n\}} f_a - \sum_{\{a|a\in A, I^2(a)=n\}} g_a f_a = v_n \quad , \quad n \in N \quad (3)$$

$$l_a \leq f_a \leq u_a \quad , \quad a \in A \quad (4)$$

(some f_a-values describe 0-1 integer investment decisions)

where the values v_n associated to the nodes $n \in N$ of graph G indicate the amounts of financial availabilities ($v_n = a_n$, $a_n > 0$, node n is a source of financing) or financial requirements ($v_n = -r_n$, $r_n > 0$, node n is a use of funds). At a node n with $v_n = 0$ the sum of in-flows equals the sum of out-flows (see FIG. 1(b) for clarification).
In (2), (3), (4) the values c_a, g_a, l_a, u_a ($a\in A$) and v_n ($n\in N$) are assumed fixed and known, but for a more realistic description of the underlying problem some of these values should be regarded as random variables.
This is especially important for the v_n ($n\in N$), which describe e.g. uncertain financial availabilities and/or requirements, uncertain cash in- & out-flows, uncertain receivables and payables etc., and less important for the lower and upper bounds l_a, u_a ($a\in A$) of the financial transactions (see GAUL (1983) for the special case with $g_a \equiv 1$ ($a\in A$) but stochastic c_a- and v_n-values).

If indices s, d and the symbol ~ are used to indicate the stochastic or deterministic data membership and the randomness of special values the following situation arises:

Assume that $\underset{\sim}{v}_n$, $n \in N_s \subset N (N_d = N \backslash N_s)$, and $\underset{\sim}{c}_a$, $a \in A_s \subset A (A_d = A \backslash A_s)$, are random variables on a given probability space $(\Omega, \mathfrak{S}, Pr)$ and notice that the financial decisions $f = (f_a \mid a \in A)$ have to be determined before the actual realizations of the random variables are known. Generally, it will not be possible to choose in advance the financial decisions f in such a way that all realizations $\underset{\sim}{v}_n(\omega)$, $\omega \in \Omega$, $n \in N_s$, can be met. Thus, there will be a nonconformity

$$\sigma_n^G(f) = \sum_{\{a \mid a \in A, I^1(a)=n\}} f_a - \sum_{\{a \mid a \in A, I^2(a)=n\}} g_a f_a \neq \underset{\sim}{v}_n(\omega), \; n \in N_s \tag{5}$$

(a violation of constraint (3)) between $\sigma_n^G(f)$ and the corresponding $\underset{\sim}{v}_n(\omega)$ which can be handled in the following sense:

If n is a source of financing and $\sigma_n^G(f) > \underset{\sim}{v}_n(\omega)$ $(\sigma_n^G(f) < \underset{\sim}{v}_n(\omega))$ more (less) than the available amount was planned for financial transactions leaving node n and the difference has to (can) be compensated e.g. by short term loan at interest rate $q_n^- > 0$ (e.g. by investment at interest yield $q_n^+ < 0$).

Similarly, if n is a use of funds (in which case $\underset{\sim}{v}_n(\omega) < 0$ holds) and $\sigma_n^G(f) > v_n(\omega)$ $(\sigma_n^G(f) < v_n(\omega))$ less (more) than the required amount was planned for financial transactions entering node n and the difference has to (can) be compensated e.g. by short term loan at interest rate $q_n^- > 0$ (e.g. by investment at interest yield $q_n^+ < 0$).
More formally these differences are handled by the expression

$$\underset{\sim}{d}_n^f(\underset{\sim}{v}_n(\omega)) = \begin{cases} q_n^+(\underset{\sim}{v}_n(\omega) - \sigma_n^G(f)) & , \\ 0 & , \quad \underset{\sim}{v}_n(\omega) \; \begin{matrix} > \\ = \\ < \end{matrix} \; \sigma_n^G(f) \\ -q_n^-(\underset{\sim}{v}_n(\omega) - \sigma_n^G(f)) & , \end{cases} \tag{6}$$

where q_n^+, q_n^- with $q_n^+ + q_n^- > 0$ are the just mentioned compensation costs (per unit of nonconformity).

Now, instead of (2), (3), (4) the following stochastic flows-with-gains problem has to be solved

$$\sum_{a \in A_d} c_a f_a + \sum_{a \in A_s} E(\underset{\sim}{c}_a) f_a + \sum_{n \in N_s} E(\underset{\sim}{d}_n^f) = \min! \tag{7}$$

$$\sum_{\{a \mid a \in A, I^1(a)=n\}} f_a - \sum_{\{a \mid a \in A, I^2(a)=n\}} g_a f_a = v_n, \quad n \in N_d \tag{8}$$

$$l_a \leq f_a \leq u_a \quad , \quad a \in A \tag{9}$$

(some f_a-values describe 0-1 integer investment decisions) where expected costs (E denotes expectation) have to be minimized under linear constraints. Generally, (7), (8), (9) is not a linear program but if one uses (or approximates the actual probability distributions by) discrete probability distributions with respect to $\underset{\sim}{v}_n$, $n \in N_s$, the above stochastic flows-with-gains problem can be solved by a sequence of non-stochastic flows-with-gains subproblems. How this can be done is described in the appendix.

Example

For ease of description the following two-assets, three-investments, four-periods example (a combination of the non-stochastic examples used in [7], [10]) is presented. FIG. 3 which already provides an optimal solution (indicated by the out-flows from the corresponding nodes) shows how a network flow description can help to clarify such a financial planning situation. Besides the out-flows only the gains are attached to the arrows of the graph (see also the difference between FIG. 1(a) and (b)), the lower, upper bounds and costs, l_a, u_a and c_a, $\underset{\sim}{c}_a$, respectively, are explained in the following text.

For simplicity, the total planning period is divided into only four time periods (of possibly different durations) and the set of different assets (distinguished e.g. by different levels of liquidity and different maturity dates) is considered to consist of only two types of assets - cash and a special near cash asset - for which transformations from one into the other are possible at positive conversion costs. For each asset four time period nodes numbered 1 up to 4 (cash area), 1^* up to 4^* (near cash asset area), respectively, are used to describe the subdivision of the total planning period. Borrowing and lending possibilities for cash (only a subset of the set of all such possibilities is considered) are indicated by the arrows between the time period nodes of the cash area with gains $g_{i,i+1}$ = 1,05 (i.e. 5% interest yield per period) and losses $g_{i+1,i} = (1,1)^{-1} \approx 0{,}909$ (i.e. 10% interest rate per period), i = 1,2,3. For these possibilities minimum cash balances $l_{i,i+1}$ = 5 UA (units of account) and upper bounds for borrowing $u_{i+1,i}$ = 10 UA, i = 1,2,3, are presupposed.

Arrows between i and i^* indicate the conversion between the two assets where losses $g_{ii^*} = g_{i^*i}$ = 0,98 (i.e. 2% costs for conversion) are taken into consideration. Besides the conversion possibility the near cash asset can be held over with gain $g_{i^*i^*+1}$ = 1,08 (i.e. 8% interest yield per period). Additionally, there is an (decision-independent)

II (initial inventory) of the near cash asset of value of 10 UA (units of account), there are (decision-independent) financial AC_1 up to AC_4 (availabilities of cash) at the time periods 1 up to 4, and, (decision-independent) financial RC_1 up to RC_4 (requirements for cash), again, at the time periods 1 up to 4. The amounts of the AC's and the RC's (some of wich are random variables indicated by the symbol ~ in which case instead of the non-stochastic amounts the expectations and the realizations together with the corresponding probabilities are given) are shown in TAB. 1.

Instead of minimizing an objective function according to (7) a maximization of the return from the final planning period is wanted in the underlying example. Therefore, an additional RFPP (return from the final planning period) node and two arrows from 4 to RFPP, 4* to RFPP, respectively, are introduced, and, the objective is

$$g_{4,RFPP}\, f_{4,RFPP} + g_{4^*,RFPP}\, f_{4^*,RFPP} - \sum_{n\in N_s} E(\tilde{d}_n^f) = \max! \qquad (10)$$

under the constraints (8), (9) which include as well investment selection possibilities to be described in the following.

For further simplicity only a set of three investment possibilities $\{I_1, I_2, I_3\}$ is considered. Two investments I_1, I_2 have four time periods lives and would start at the beginning of the first time period, the remaining I_3 with a three time periods life would start at the beginning of the second time period, thus, all investments end at the final planning period. TAB. 2 describes the different investment possibilities by their cash flows.

For the performance of the different investment possibilities a LC (line of credit) of amount of 25 UA could be used. Notice, that three SU_1 up to SU_3 (supplementary) nodes are needed to distinguish between borrowing from the LC and those financial transactions using cash from the first (or second (for I_3)) time period.

Introducing investment nodes (labelled I_i, i=1,2,3) and an ID (investment decision) node the optimal investment selection can be described as follows: There are 3 arrows leaving the ID node and 2^3 investment combinations according to the number of subsets of $\{I_1, I_2, I_3\}$. The outflows from the ID-node establish the investment selection and are described by a three-dimensional 0-1 integer valued investment decision vector where (0,0,0) means "no investment" and e.g. (0,1,1) means "investment I_2 and I_3". Thus, in general, a branch & bound procedure should be performed to reject unfavourable or infeasible investment combinations.

time period	1	2	3	4	1	2	3	4
	AC 1	AC 2	$\tilde{AC}$ 3	$\tilde{AC}$ 4	RC 1	RC 2	RC 3	RC 4
fixed value or expectation	5.00	10.00	20.00	15.00	16.00	12.00	8.00	8.00
realizations			11.00	9.00				
			13.00	10.00				
			15.00	12.00				
			17.00	13.00				
			20.00	15.00				
			22.00	16.00				
			24.00	18.00				
			26.00	19.00				
			29.00	21.00				
probabilities			.006	.006				
			.021	.021				
			.068	.082				
			.216	.202				
			.320	.290				
			.209	.242				
			.112	.118				
			.042	.033				
			.006	.006				

TAB. 1: Financial Availabilities of/Requirements for cash (in UA (Units of Account))

time period	1	2	3	4
investments				
I_1	− 10	5	5	5
I_2	− 12	6	6	6
I_3		− 10	7	5

TAB. 2: Cash-flows of Investments (in UA(Units of Account))

FIG. 2 shows the different investment decisions together with the corresponding values of the objective function (10) and indicates optimality or infeasibility.

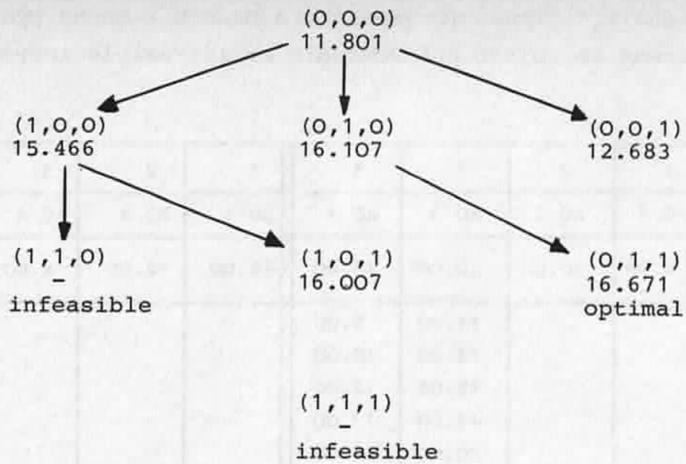

FIG. 2: Investment Selection

In the appendix (see (15),(16),(17),(18),(19)) it is explained how the flows-with-gains approach can be used to enforce that a decision for (against) investment I_i causes (hinders) the corresponding cash flows to flow. Notice, that the cash flow values correspond to the gains attached to the arrows outgoing from the single investment nodes (and notice, too, that a negative gain creates a reversal of the flow direction according to (1)).

Lastly, the stochastic nature of some of the data of such financial planning problems has to be rementioned. Here, for simplicity, it is assumed that only the $\underset{\sim}{AC}_3$, $\underset{\sim}{AC}_4$ are random variables. Using discrete distributions as given in TAB. 1 and choosing two subsequent realizations for each random variable a deterministic subproblem of the flows-with-gains type is solved, and, it is checked whether this subproblem already creates an optimal solution for the original problem or whether new subsequent realizations of the random variables have to be determined to yield improvements. After finite many steps the algorithm terminates with optimality or infeasibility (see the appendix for a more formal description). CPU-time for the underlying example amounted to about 12 seconds for several test runs.

The stochastic flows-with-gains approach behaves as it should do. It detects infeasible investment combinations (which would violate restrictions imposed by the financial availabilities of/requirements for cash and the upper bounds of borrowing and the line of credit, etc., and it determines an optimal financial decision vector $f = (f_a | a \in A)$ according to the underlying costs- and gains-structure.

As can be seen from FIG. 3 in the optimal solution all initial inventory of the near cash asset and a short term loan of 6.82 UA is needed in the first time period of the cash area to meet the restrictions imposed by the requirements for cash and the minimum cash balance. In the second time period of the cash area the maximum amount of 10 UA of short term loan is used to meet the minimum cash balance, to repay the short term loan of the time period before the actual one and to finance 6.521 UA of investment I_3. In this second time period the first cash in-flow of investment I_2 (which was totally financed by the line of credit) of 6 UA arrives. In the third time period no further short term loan is needed, instead, 10.25 UA of cash can be converted to the near cash asset area where a higher interest yield can be achieved. Notice, that also in the final time period of the cash area all cash not needed for adjusting the line of credit is converted to the near cash asset area as no minimum cash balance is presupposed for the financial transaction to the RFPP node.

Notice, too, that the amount of 20.182 UA for financing the investment decision does not utilize the whole line of credit of 25 UA for which losses of 0,751 for investments I_1, I_2 and 0,826 for investment I_3 have to be taken into consideration.

Finally, the planned return from the final planning periods amounts to $11.895 \cdot 1.08 = 12.8466$ UA whereas the optimal value of the objective function of the form (10) is 16.671 UA as indicated in FIG. 2. This pleasing positive difference is due to the cautious planning of the in-flows of 15 UA, 13 UA from $\underset{\sim}{AC}_3$, $\underset{\sim}{AC}_4$ (the expectations are $E(\underset{\sim}{AC}_3) = 20$, $E(\underset{\sim}{AC}_4) = 15$) indicating an additional expected return which depends on the chosen costs of compensation $q_n^- = 1,24$ ($q_n^- = 1.12$) for short term loan used in time period 3 (time period 4), $q_n^+ = -1,14$ ($q_n^+ = -1.05$) for short term investment in time period 3 (time period 4) where n, here, denotes the $\underset{\sim}{AC}_3$ node (the $\underset{\sim}{AC}_4$ node), respectively.

Conclusion

A network flow formulation together with a branch & bound approach was used for the description of a typical example from the area of financial planning. To take into consideration the uncertainty element related to most financial planning decisions tools from graph theory and stochastic programming have been combined to a stochastic flows-with-gains approach. Advantageously, the stochastic flows-with-gains formulation

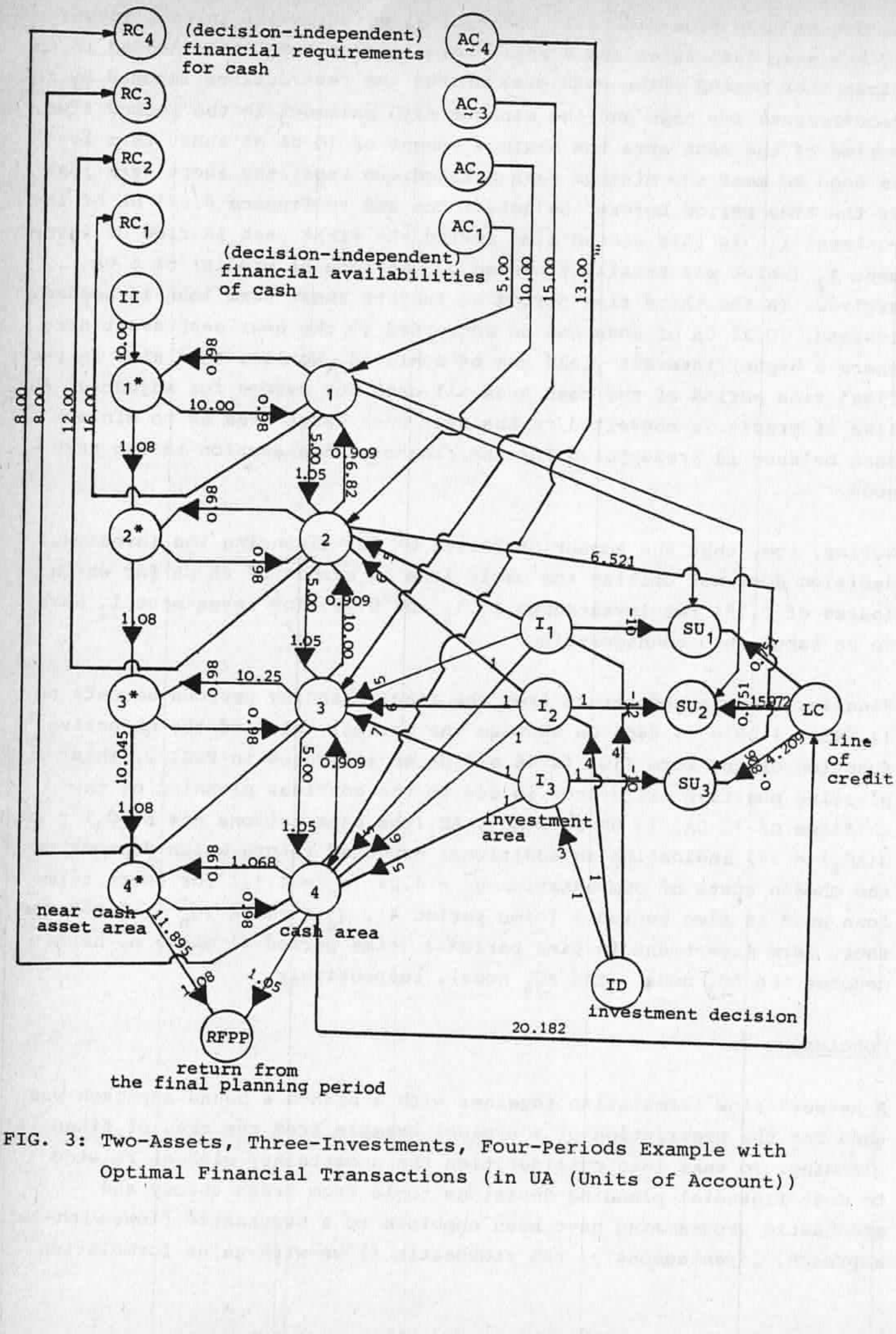

FIG. 3: Two-Assets, Three-Investments, Four-Periods Example with Optimal Financial Transactions (in UA (Units of Account))

of the financial planning problem was solved by a sequence of non-stochastic flows-with-gains financial planning subproblems (for which solution procedures as given e.g. in [7], [10] could be used). If a more thoroughly structured total planning period is needed a multiple-period unequal-period periodically recomputed schedule as given in GAUL (1983) can be formulated and solved within the stochastic flows-with-gains approach without difficulty.

Appendix

If $v_{n\varkappa}$, $\varkappa = 1,\dots,\varkappa_n$, with $v_{n1} < v_{n2} < \dots < v_{n\varkappa_n}$ denote the realizations of the finite discrete distributed random variables $\underset{\sim}{v}_n$, $n \in N_s$, the stochastic flows-with-gains problem (7), (8), (9) can be rewritten as a linear program of the form

$$\sum_{a\in A_d} c_a f_a + \sum_{a\in A_s} E(\underset{\sim}{c}_a) f_a + \sum_{n\in N_s} \sum_{\varkappa=1}^{\varkappa_n} (q_n^+ y_{n\varkappa}^+ + q_n^- y_{n\varkappa}^-) \Pr(\underset{\sim}{v}_n = v_{n\varkappa}) = \min!$$

$$\sum_{\{a|a\in A, I^1(a)=n\}} f_a - \sum_{\{a|a\in A, I^2(a)=n\}} g_a f_a = v_n \quad , \quad n \in N_d \qquad (11)$$

$$\sigma_n^G(f) + y_{n\varkappa}^+ - y_{n\varkappa}^- = v_{n\varkappa} \quad , \quad n \in N_s, \ \varkappa=1,\dots,\varkappa_n \qquad (12)$$

$$l_a \le f_a \le u_a \quad , \quad a \in A \qquad (13)$$

$$y_{n\varkappa}^+ \ge 0, \quad y_{n\varkappa}^- \ge 0 \quad , \quad n \in N_s, \ \varkappa=1,\dots,\varkappa_n \qquad (14)$$

(some f_a-values describe 0-1 integer investment decisions)

where $y_{n\varkappa}^+$, $y_{n\varkappa}^-$ describe the nonconformity according to (5), and, for which - dependent on the numbers $\varkappa_n$ of the realizations of the random variables $\underset{\sim}{v}_n$, $n \in N_s$ - the dimensions could grow too large (at least when using approximation arguments) to be handled by conventional LP methods.

Supposed, there is a subdivision of the total planning period into T time periods (subperiods of possibly different durations) then, at least, T time period nodes are needed to create a cash area in which cash in- & out-flows dependent on these different time periods can be distinguished.

If, additionally, $\{I_1,\dots,I_m\}$ is a set of interesting investment possibilities, and, if each I_i is described by its sequence of periodic cash flows then, at least, m investment nodes and arrows from each investment node to those time period nodes in the cash area for which non-zero cash flows exist are needed to affix an area of investment decisions to the cash area (see FIG. 3).

Introducing an ID (investment decision) node the selection of an optimal combination of investment possibilities can be performed by a branch & bound procedure by computing an m-dimensional investment decision vector of 0-1 integer values $IN = (f_{ID,I_i} \mid i \in \{1,\dots,m\})$ with

$$f_{ID,I_i} = \begin{cases} 1, & \text{investment } I_i \text{ is selected} \\ 0, & \text{otherwise} \end{cases} \quad , \; i \in \{1,\dots,m\} \qquad (15)$$

rejecting investment combinations which would violate financial constraints or which have a less preferable value of the objective function of the underlying mathematical problem. If

g_{ID,I_i} corresponds to the number of non-zero cash flows of I_i (16)

the following constraints (a subset of the constraints of the form (11), (13))

$$\sum_{t \in \{1,\dots,T\}} f_{I_i,t} - g_{ID,I_i} f_{ID,I_i} = 0 \quad , \quad I_i \in \{I_1,\dots,I_m\} \qquad (17)$$

$$0 \le f_{I_i,t} \le 1$$

force

$$f_{ID,I_i} = \begin{cases} 1 \\ 0 \end{cases} \Longrightarrow f_{I_i,t} = \begin{cases} 1 & \text{for all } t \in \{1,\dots,T\} \text{ for which} \\ 0 & \text{non-zero cash flows exist} \end{cases} \qquad (18)$$

If, now,

$g_{I_i,t}$ corresponds to the cash flow of I_i for time period t (19)

the $g_{I_i,t}\, f_{I_i,t}$ in-flows (notice, that a negative gain value creates a reversal of the flow direction according to (1)) describe the periodic cash flows of I_i, $i \in \{1,\dots,m\}$ (see FIG. 3 where additional supplementary nodes are needed to distinguish between possible borrowing activities from a line of credit and financial transactions which use cash from the corresponding time periods out of the cash area).

Additional nodes for (decision-independent) financial availabilities and requirements of cash and different types of assets can be incorporated as well as additional arrows for the appropriate financial transactions but for space limitations a discussion of such possibilities as given in the example has to be sufficient.

To avoid difficulties with the solution of the stochastic flows-with-gains problem of the form (11), (12), (13), (14) one can proceed as follows:

First, select the investment decision vector $IN = (f_{ID,I_i} \mid i \in \{1,\dots,m\})$.

Second, select, for $n \in N_s$, subsequent realizations v_{nx_n}, $v_{n(x_n+1)}$ and define a realization index vector $x = (x_n \mid n \in N_s)$.

Third, create a new graph $G^x = (N^x, A^x, I^x)$ and solve the non-stochastic flows-with-gains problem (of nearly the same size as the non-stochastic problem (2), (3), (4))

$$\sum_{a \in A^x} c_a^x f_a$$

$$\sum_{\{a \mid a \in A^x, I^{x1}(a)=n\}} f_a - \sum_{\{a \mid a \in A^x, I^{x2}(a)=n\}} g_a f_a = 0 \quad , n \in N^x \qquad (20)$$

$$l_a^x \le f_a \le n_a^x \quad , a \in A^x \qquad (21)$$

In CLEEF/GAUL (1980), GAUL (1983) the construction of G^x and $(c_a^x, l_a^x, u_a^x \mid a \in A^x)$ is described. The optimal solution f of (20), (21) shows how an optimal solution for (11), (12), (13), (14) has to look like if for certain optimal dual values (of the dual program to (20), (21)) α_n^x, β_n^x attached to the stochastic counterpart nodes $n \in N_s^x$ $(=N_s)$ the following condition

$$\alpha_n^x \le (q_n^+ + q_n^-)\ \Pr(\underset{\sim}{v}_n = v_{nx_n}) \qquad (22a)$$

$$, n \in N_s$$

$$\beta_n^x \le (q_n^+ + q_n^-)\ \Pr(\underset{\sim}{v}_n = v_{n(x_n+1)}) \qquad (22b)$$

is valid. Otherwise the realization index vector is changed according to

$$x_n^{new} = x_n^{old} + \begin{cases} 1 \ , & n \text{ for which (22a) is not valid} \\ (-1), & n \text{ for which (22b) is not valid} \\ 0 \ , & \text{otherwise} \end{cases} \qquad (23)$$

After finite many alterations of the form (23) an optimal solution f_{IN} of (20), (21) dependent on the investment decision vector IN is determined.

A branch & bound procedure with respect to IN gives a global optimal solution.

Acknowledgement

Thanks are due to D. Baier, M. Both, B. Glaser who adapted the existing computer program for [9] to the underlying financial planning situation.

References

[1] ALBACH, H.: Das optimale Investitionsbudget bei Unsicherheit, Zeitschrift für Betriebswirtschaft 37 (1967) 503 - 518.

[2] BRICK, I.E./MELLON, W.G./SURKIS, J./MOHL, M.: Optimal Capital Structure: A Multi-Period Programming Model for Use in Financial Planning, J. Banking & Finance 7 (1983) 45 - 67.

[3] BÜHLER, W.: Flexible Investitions- und Finanzplanung bei unvollkommen bekannten Übergangswahrscheinlichkeiten, OR-Spektrum 2 (1981) 207 - 221.

[4] CHRISTOFIDES, N./HEWINS, R.D./SALKIN, G.R.: Graph Theoretic Approaches to Foreign Exchange Operations, J. of Financial & Quantitative Analysis XIV (1979) 481 - 500.

[5] CLEEF, H.J./GAUL, W.: A Stochastic Flow Problem, J. Information & Optimization Sc. 1 (1980) 229 - 270.

[6] COHEN, K.J./MAIER, S.F./VAN DER WEIDE, J.H.: Recent Developments in Management Science in Banking, Management Sc. 27 (1981) 1097 - 1119.

[7] CRUM, R.L./KLINGMAN, D.D./TAVIS, L.A.: Implementation of Large-Scale Financial Planning Models: Solution Efficient Transformations, J. of Financial & Quantitative Analysis XIV (1979) 137 - 152.

[8] DENK, R.: Kurzfristige Finanzdisposition mit Hilfe von Transportmodellen, Zeitschrift für Betriebswirtschaft 48 (1978) 356 - 367.

[9] GAUL, W.: Short Term Financial Planning under Uncertainty within a Graphtheoretical Framework, Discussion-paper 46, Institut für Entscheidungstheorie und Unternehmensforschung, Universität Karlsruhe (1983), modified version of a paper presented at the 2. Symposium of Money, Banking and Insurance, Karlsruhe (December 1982).

[10] GOLDEN, B./LIBERATORE, M./LIEBERMAN, C.: Models and Solution Techniques for Cash Flow Management, Comput. & Ops. Res. 6 (1979) 13 - 20.

[11] GREGORY, G.: Cash Flow Models: A Review, Omega 4(1976) 643 - 656.

[12] GRINOLD, R.C.: Calculating Maximal Flows in a Network with Positive Gains, Operations Research 21 (1973) 528 - 541.

[13] HORST, R.: Devisenarbitrage als Flußprobleme, Zeitschrift für Operations Researcn 19 (1975a) 73 - 87.

[14] HORST, R.: Maximale Netzwerkflüsse mit Kantengewinnen, Operations Research Verfahren (Methods of Operations Research) XXI (1975b) 113 - 138.

[15] JARVIS, J.J./JERZIOR, A.M.: Maximal Flow with Gains through a Special Network, Operations Research 20 (1972) 678 - 688.

[16] JEWELL, W.S.: Optimal Flow through Networks with Gains, Operations Research 10 (1962) 476 - 499.

[17] JOHNSON, E.L.: Networks and Basis Solutions, Operations Research 14 (1966) 619 - 623.

[18] KALL, P.: Stochastic Linear Programming, Springer (1976).

[19] KALLBERG, J.G./White, R.W./ZIEMBA, W.T.: Short Term Financial Planning under Uncertainty, Management Sc. 28 (1982) 670 - 682.

[20] LEVY, H./SARNAT, M. (Eds.): Financial Decision Making under Uncertainty, Academic Press (1977).

[21] MAYEDA, W./VAN VALKENBURG, M.E.: Properties of Lossy Communication Nets, IEEE Transaction on Circuit Theory 12 (1965) 334 - 338.

[22] MINIEKA, E.: Optimal Flow in a Network with Gains, Infor. Can. J. Oper. Res. 10 (1972) 171 - 178.

[23] ONAGA, K.: Dynamic Programming of Optimum Flows in Lossy Communication Nets, IEEE Transactions on Circuit Theory 13 (1966) 282 - 287.

[24] ONAGA, K.: Optimum Flows in General Communication Networks, J. Franklin Institute 283 (1967) 308 - 327.

[25] ORGLER, Y.E.: Cash Management: Methods and Models, Wadsworth (1970).

[26] RUTENBERG, D.P.: Maneuvering Liquid Assets in a Multi-National Company: Formulation and Deterministic Solution Procedures, Management Sc. 16 (1970) B-671 - B-684.

[27] SARTORIS, W.L./HILL, N.C.: A Generalized Cash Flow Approach to Short-Term Financial Decisions, J. of Finance XXXVIII (1983) 349 - 360.

[28] SRINIVASAN, V.: A Transshipment Model for Cash Management Decisions, Management Sc. 20 (1974) 1350 - 1363.

[29] STANCU-MINASIAN, I.M./WETS, M.J.: A Research Bibliography in Stochastic Programming, Operations Research 24 (1976) 1078 - 1119.

[30] TRUEMPER, K.: On Max Flows with Gains and Pure Min-Cost Flows, SIAM J. Appl. Math. 32 (1977) 450 - 456.

[31] WEINGARTNER, M.: Mathematical Programming and the Analysis of Capital Budgeting Problems, Princeton University Press (1963).

[32] ZIEMBA, W.T./VICKSON, R.G. (Eds): Stochastic Optimization Models in Finance, Academic Press (1975).