Annals of Discrete Mathematics 28 (1985) 107-124

RELIABILITY-ESTIMATION IN STOCHASTIC GRAPHS WITH TIME-ASSOCIATED ARC-SET RELIABILITY PERFORMANCE PROCESSES

Wolfgang GAUL

University of Karlsruhe (TH), West Germany

In this paper situations are considered in which the reliability-behaviour of the arcs of stochastic graphs is described by time-associated reliability performance processes.

Reliability-estimations, e.g. lower (and for special cases upper) reliability bounds — additional to the known minimal cut lower bound of Esary and Proschan — are yielded by a proper decomposition of the underlying stochastic graph. This decomposition allows a successive determination of the reliability estimation by using reliability estimations of stochastic subgraphs which should be of interest when the underlying stochastic graph is large.

Comparisons of the different bounds are made within an example of simplest form.

1. Introduction

There are some interesting directions concerning stochastic aspects within application-relevant graphtheoretical problems, one of them consists in stochastic programming on graphs, see e.g. Cleef and Gaul [6], [7], another in models of the reliability-behaviour of graphs.

In most papers dealing with reliability problems in stochastic graphs the model description is done from a static stochastic viewpoint allowing that the elements of the graph (nodes, arcs) can take only two states — functioning or having failed — with probabilities independent of time.

Graphtheoretical reliability measures depend on an appropriate connectivity notation (well-known measures are given e.g. by the probability that a specified pair of nodes or a specified subset of nodes belong to a "connected functioning subgraph," a notation which has to be defined according to the fact whether directed or undirected graphs are used in which nodes and/or arcs are subject to random failure).

Of course, the simpler the structure of the underlying stochastic graph is the more realistic stochastic descriptions are possible (see e.g. Barlow and Proschan [1], where – if the underlying stochastic graph is a path with reliable nodes (a series-system built by the arcs of the path) – tools from availability theory in connection with renewal theory can be applied).

However, standard reliability problems in stochastic graphs don't use time-associated reliability performance processes for model description as will be assumed throughout the rest of this paper, thus, literature concerning some known directions of reliability graph problems is not explicitly mentioned here but see Frank and Gaul [14] (where connectedness probabilities in stochastic graphs with randomly failing nodes and arcs are considered), Gaul and Hartung [18] (where bounding distribution functions are computed when the arcs of the underlying stochastic graph can take several states of reliability between complete failure and perfect functioning, see also Barlow and Wu [3], El-Neweihi, Proschan and Sethuraman [9] for the first description of multistate reliability models) and the references cited there.

With respect to the dependence structure of the random variables used for model description of reliability problems association (which was first mentioned in Esary, Proschan and Walkup [13], a more recent paper is Jogdeo [20]) can be used. Weakening the usual and restrictive independence assumption to the case of time-associated performance processes was done in Esary and Proschan [12] and is adopted here to derive estimators, e.g. lower (and for special cases upper) bounds for the nodebasis – nodecontrabasis (see Harary, Norman and Cartwright [19] for graphtheoretical notations) connectedness probability in stochastic acyclic digraphs – additional to the known minimal cut lower bound of Esary and Proschan.

This bound was first established in Esary and Proschan [11] and improved by Bodin [4] using modular decompositions (for the use of modules which are also of interest in fields other than reliability theory see e.g. Butterworth [5]), its generalization to the time-associated case was given in the already mentioned paper by Esary and Proschan [12].

In this paper reliability-bounds are constructed by means of a proper decomposition of the underlying stochastic acyclic digraph first described in Gaul [16] for project digraphs.

Dependent on the used decomposition improvements of some of the bounds (including those of Esary and Proschan) can be obtained. Comparisons of the different bounds are made within an example of simplest form.

2. Problem formulation

For ease of description some of the frequently used graphtheoretical notations are given in the following, for more detailed and additional explanations see e.g. Harary, Norman and Cartwright [19].

Let $D=(N, A, f)$ describe a digraph where $N \neq \emptyset$ denotes the set of nodes, A the set of arcs, $f=(f^1, f^2)$ with $f^i: A \to N$, $i=1, 2$, the incidence mapping with $f^1(a), f^2(a)$ as starting-, end-node of $a \in A$. For abbreviation, sometimes, only D is written for a digraph in which case $N(D)$, $A(D)$ is used to denote the nodes, arcs of D. The incidence mapping is mostly omitted. In this case the tupel $(N(D), A(D))$ is used instead of D.

For two digraphs D_i, $i=1, 2$, call

$$D_1 \subset D_2 \text{ (subdigraph)} \quad \text{iff } N(D_1) \subset N(D_2),\ A(D_1) \subset A(D_2),$$

$$D_1 \overset{\cup}{\cap} D_2 \text{ (union, intersection digraph) iff } N(D_1 \overset{\cup}{\cap} D_2) = N(D_1) \overset{\cup}{\cap} N(D_2),$$
$$A(D_1 \overset{\cup}{\cap} D_2) = A(D_1) \overset{\cup}{\cap} A(D_2).$$

In the following it suffices to consider only gsp (*g*eneralized *s*eries-*p*arallel)-digraphs of the form

$$D_{pq} = (N(D_{pq}), A(D_{pq}))$$

which are finite, acyclic, weakly connected directed graphs the nodebasis (node-contrabasis) of which consists of the single node $p \in N(D_{pq})$ $(q \in N(D_{pq}))$. gsp-digraphs are of importance because they generalize the description of two terminal series-parallel systems which obviously can be represented by gsp-digraphs. They also enclose project digraphs (when parallel arcs are not allowed) structural properties of which have been described in Gaul [16] and can be used in the following context.

One question is whether for a given pair of nodes $i, j \in N(D_{pq})$ there exists a gsp-subdigraph $D_{ij} \subset D_{pq}$. If this is the case, special gsp-subdigraphs of interest are the maximal gsp-subdigraph denoted by $\hat{D}_{ij}$ and the minimal gsp-subdigraphs denoted by P_{ij} and called paths from i to j. $(P_{ij})_k$, $(P_{ij})^k$, $k \in N(P_{ij})$, gives the subpath of P_{ij} from i to k, k to j, respectively. The set of paths belonging to D_{ij} is denoted by $P(D_{ij})$.

For a gsp-digraph D_{pq} there exists a bijective mapping called (ascending) level-assignment $l : N(D_{pq}) \to \{0, 1, \dots, m\}$ $(m = |N(D_{pq})| - 1)$ with

$$a \in A \ \Rightarrow\ l(f^1(a)) < l(f^2(a)) \quad (\text{and } l(p)=0,\ l(q)=m).$$

With the identification $N(D_{pq}) := \{0, 1, \ldots, m\}$ the nodes of D_{pq} are assumed to be topologically ordered according to such a level assignment (this assumption is needed for the successive determination of the reliability estimation), and, from now on

$$\text{the notation } \hat{D}_{0m} \text{ is used instead of } D_{pq}. \tag{1}$$

$N(\hat{D}_{0m})$ is assumed to consist of perfect nodes only (which do never fail). $A(\hat{D}_{0m})$ consists of unreliable elements the reliability behaviour of which is described by the (vector) reliability performance process

$$\{X(t), t \in T\} = \{(X_a(t), a \in A(\hat{D}_{0m})), t \in T\},$$

where $X_a(t)$ are Bernoulli-distributed random variables on a given probability space $(\Omega, G, \Pr)$ with

$$X_a(t) = \begin{cases} 1 & \text{arc } a \text{ is functioning} \\ & \text{at time } t, \\ 0 & \text{otherwise}, \end{cases} \qquad a \in A(\hat{D}_{0m}),\ t \in T \subset [0, \infty). \tag{2}$$

For fixed $\omega \in \Omega$ the sample functions $X_a(t, \omega)$ are assumed to be continuous from the right on T.

Now, for fixed time $t \in T$, the stochastic graph is described by the tupel

$$(N(\hat{D}_{0m}), A(\hat{D}_{0m}), f, (X_a(t), a \in A(\hat{D}_{0m})))$$

but, again, at least for mainly graphtheoretical considerations the incidence mapping and the reliability performance processes are omitted.

With respect to the dependence structure of the random variables describing the reliability-behaviour it is presupposed that the reliability performance process is time-associated, which means that for all finite sets of times $T_k = \{t_1, \ldots, t_k\} \subset T$

$$\{X_a(t), t \in T_k, a \in A(\hat{D}_{0m})\} \text{ is a set of associated random variables}$$

(see [2], [12], [13], [20] for properties of association or/and time-association and the discussion of special cases as independence and positively total dependence and a variety of maintenance situations).

Of course, for two nodes $i, j \in N(\hat{D}_{0m})$ for which gsp-subdigraphs D_{ij} exist, an interesting question is whether there will be a functioning path P_{ij} (a path with functioning arcs) from i to j.

More formally, for every gsp-subdigraph D_{ij} a so-called structure function

$$\varphi_{D_{ij}}=\begin{cases}1 & \text{there exists a functioning} \\ & \text{path } P_{ij}\in P(D_{ij}), \\ 0 & \text{otherwise},\end{cases}$$

can be defined with its path-representation

$$\varphi_{D_{ij}}(X(t))=1-\prod_{P_{ij}\in P(D_{ij})}(1-\prod_{a\in A(P_{ij})}X_a(t))=\max_{P_{ij}\in P(D_{ij})}\{\prod_{a\in A(P_{ij})}X_a(t)\}. \quad (3)$$

Using the minimal cuts of D_{ij} – a minimal cut $C_{ij}\subset A(D_{ij})$ is a minimal set of arcs with the property $C_{ij}\cap A(P_{ij})\neq\emptyset$, $\forall P_{ij}\in P(D_{ij})$ – the following cut representation

$$\varphi_{D_{ij}}(X(t))=\prod_{C_{ij}\in C(D_{ij})}\varphi_{C_{ij}}(X(t)) \quad (\text{with } \varphi_{C_{ij}}(X(t))=1-\prod_{a\in C_{ij}}(1-X_a(t)) \quad (4)$$

is equivalent to (3) where $C(D_{ij})$ denotes the set of cuts of D_{ij}.

For fixed time $t\in T$ the structure function $\varphi_{D_{ij}}$ is a binary non-decreasing function with $\varphi_{D_{ij}}(0,\ldots,0)=0$, $\varphi_{D_{ij}}(1,\ldots,1)=1$.

For fixed $\omega\in\Omega$ the sample function $\varphi_{D_{ij}}(X(t,\omega))$ is continuous from the right on T.

$$R_{D_{ij}}(\tau)=\Pr(\varphi_{D_{ij}}(X(t))=1, \quad \forall t\in T(\tau)) \quad (\text{with } T(\tau)=[0,\tau]\cap T) \quad (5)$$

is an intuitive reliability measure for a gsp-digraph D_{ij} but for larger and more complex structured graphs the determination of the exact value of (5) can be difficult.

In this situation one can calculate the minimal cut lower bound of Esary and Proschan

$$EP_{D_{ij}}(\tau)=\prod_{C_{ij}\in C(D_{ij})}R_{C_{ij}}(\tau) \quad (6)$$

(with $R_{C_{ij}}(\tau)=\Pr(\varphi_{C_{ij}}(X(t))=1, \forall t\in T(\tau))$) for which

$$\{X(t), t\in T\} \text{ time-associated} \Rightarrow R_{D_{ij}}(\tau)\geqslant EP_{D_{ij}}(\tau) \quad (7)$$

is valid, and, of course, additional bounding possibilities for the reliability estimation would be of interest.

3. gsp-digraph decomposition

With respect to the given gsp-digraph $\hat{D}_{0m}$ the following notations are useful: Let be

$$\delta_n = \{D_{in},\ D_{in} \text{ is gsp-subdigraph, } i<n\}$$

a system of gsp-subdigraphs which all have the same nodecontrabasis $n \in N(\hat{D}_{0m})$, $1 \leqslant n \leqslant m$,

$$B(\delta_n) = \{i,\ i \in N(\hat{D}_{0m}),\ D_{in} \in \delta_n\}$$

the set of the nodebasis-nodes of the gsp-subdigraphs of δ_n.

Call δ_n proper if

$$\forall D^1_{i_1 n}, D^2_{i_2 n} \in \delta_n:\ D^1_{i_1 n} \cap D^2_{i_2 n} = \begin{cases} (\{i, n\}, \emptyset) & i_1 = i_2 = i, \\ (\{n\}, \emptyset) & \text{otherwise}, \end{cases} \tag{8}$$

$$\forall P_{0n} \subset \hat{D}_{0m}\quad \exists\, i \in B(\delta_n) \cap N(\hat{D}_{0m}),\quad D_{in} \in \delta_n:$$

$$P_{0n} = (P_{0n})_i \cup (P_{0n})^i \quad \text{with} \quad \begin{matrix} (P_{0n})_i \cap \delta_n \subset (B(\delta_n), \emptyset) \\ (P_{0n})^i \in P(D_{in}). \end{matrix} \tag{9}$$

Such proper gsp-subdigraph systems always exist, e.g. $\delta_n = \{\hat{D}_{0n}\}$ is proper. Because of (8) the gsp-subdigraphs of the proper δ_n are arc-disjoint and node-disjoint except for the common nodecontrabasis n and, eventually, a common nodebasis i, (9) establishes a relation between $P(\hat{D}_{0n})$ and δ_n. For properties of proper systems of project digraphs see Gaul [16], the following theorem (for the proof of which see Gaul and Hartung [18] in the more general multistate reliability framework) gives a hint why proper systems of gsp-subdigraphs could be useful.

Theorem 1. *If* $\delta_n = \{D_{in}\}$ *is a proper gsp-subdigraph system then*

$$\varphi_{\hat{D}_{0n}} = 1 - \prod_{D_{in} \in \delta_n} (1 - \varphi_{\hat{D}_{0i}}\, \varphi_{D_{in}}) = \max_{D_{in} \in \delta_n} \{\varphi_{\hat{D}_{0i}}\, \varphi_{D_{in}}\}.$$

From Theorem 1 it follows that a successive determination of the structure function (for $n=m$ one gets the structure function of the underlying gsp-digraph $\hat{D}_{0m}$) is possible which depends on the chosen proper gsp-subdigraph system δ_n.

Among different proper gsp-subdigraph systems the following relation will be of interest:

If $\delta_n=\{D_{ln}\}$, $\delta_n^*=\{D_{l^*n}\}$ are two proper gsp-subdigraph systems with $A(\delta_n)\supset A(\delta_n^*)$ then there exists a proper gsp-subdigraph system $\delta_n^{**}=\{D_{l^{**}n}\}$ with

$$\begin{aligned}&\text{(i) } A(\delta_n^{**})=A(\delta_n),\\ &\text{(ii) } \forall D_{i^*n}^*\in\delta_n^* \;\; \exists\, D_{i^{**}n}^{**}\in\delta_n^{**}: D_{i^*n}^*\subset D_{i^{**}n}^{**}.\end{aligned} \tag{10}$$

For δ_n^*, δ_n^{**} fulfilling (10) the notation $\delta_n^*\Subset\delta_n^{**}$ is used.

4. Reliability estimation via lower (upper) bounds

The results of the following lemmas are needed.

Lemma 1.

$$\alpha_{ik}\in[0,1],\qquad \begin{matrix} i=1,\ldots,r\\ k=1,\ldots,s\end{matrix}\;\Rightarrow$$

$$\text{(i) } \max_{1\leqslant i\leqslant r}\{\prod_{k=1}^{s}\alpha_{ik}\}\leqslant\prod_{k=1}^{s}\max_{1\leqslant i\leqslant r}\{\alpha_{ik}\},$$

$$\text{(ii) } \max_{1\leqslant i\leqslant r}\{\prod_{k=1}^{s}\alpha_{ik}\}\leqslant 1-\prod_{i=1}^{r}(1-\prod_{k=1}^{s}\alpha_{ik}),$$

$$\alpha_{ik}\in\{0,1\},\qquad \begin{matrix} i=1,\ldots,r,\\ k=1,\ldots,s,\end{matrix}\qquad \textit{additional } \alpha_{ik_1}\geqslant\alpha_{ik_2},\; k_1\leqslant k_2,\;\forall i\;\Rightarrow$$

$$\text{(iii) } \max_{1\leqslant i\leqslant r}\{\prod_{k=1}^{s}\alpha_{ik}\}=\prod_{k=1}^{s}\max_{1\leqslant i\leqslant r}\{\alpha_{ik}\}=1-\prod_{i=1}^{r}(1-\prod_{k=1}^{s}\alpha_{ik}).$$

Proof.

(i) is obvious for less restrictive assumptions.

(ii) follows with $\gamma_i=\prod_{k=1}^{s}\alpha_{ik}$ by induction with respect to r.

(iii) Nothing has to be shown if $\max_{1\leqslant i\leqslant r}\{\alpha_{ik_0}\}=0$ for some $k_0\in\{1,\ldots,s\}$, but $\max_{1\leqslant i\leqslant r}\{\alpha_{ik}\}=1$, $\forall k$, causes the existence of $i_0\in\{1,\ldots,r\}$ with $\alpha_{i_0s}=1$ and because of the non-increasing property $\alpha_{i_0k}=1$, $\forall k$. □

With the definition (see Esary and Marshall [10])

> a device with reliability performance process $\{X(t), t \in T\}$ has a life in $T(\tau)$ if $\Pr(X(t)=1, \forall t \in [0, s) | X(s)=1) = 1$ for all $s \in T(\tau)$ with $\Pr(X(s)=1)>0$ holds,

one has

Lemma 2. *If D is a* gsp-*digraph with life in* $T(\tau)$ *then*

$$\varphi_D(X(t_1)) \geqslant \varphi_D(X(t_2)), \quad t_1 < t_2, \quad t_1, t_2 \in T(\tau).$$

Proof. Let be

$$A=\{\omega | \varphi_D(X(t_2,\omega))=1\} \text{ with } \Pr(A)>0, \tilde{A}=\{\omega | \varphi_D(X(t_1,\omega))<\varphi_D(X(t_2,\omega))\}.$$

One has $\tilde{A} \subset A$ because φ_D is a binary function. From $\Pr(\tilde{A})>0$ the contradiction

$$\Pr(\varphi_D(X(t))=1, \forall t \in [0, t_2) | \varphi_D(X(t_2))=1)$$
$$\leqslant \Pr(\varphi_D(X(t_1))=1 | \varphi_D(X(t_2))=1)=\frac{\Pr(A \backslash \tilde{A})}{\Pr(A)}<1$$

follows. □

Remark. It is Lemma 1(i) which always allows to get lower reliability bounds. Lemma 1(ii) indicates a possibility which can yield improvements for lower bounds but only if special gsp-subdigraphs have lives in $T(\tau)$ upper reliability bounds are obtainable according to Lemma 1(iii) by using the non-increasing property of Lemma 2.

For the proofs of the following theorems notice that if

$$\begin{aligned} &E \subset T(\tau) \text{ is a countable, dense subset of } T(\tau),\\ &T_k=\{t_1, \ldots, t_k\} \subset E \text{ is a finite subset of } E \text{ with } T_k \nearrow E \, (k \to \infty),\\ &D, D_1, D_2 \text{ are gsp-subdigraphs of a proper } \delta\text{-system,} \end{aligned} \tag{11}$$

then

$$\Pr(\varphi_D(X(t))=1, \; t \in T_k) \searrow \Pr(\varphi_D(X(t))=1, \; \forall t \in E) \quad (k \to \infty) \tag{12}$$

by monotone convergence, and,

$$R_D(\tau)=\Pr(\varphi_D(X(t))=1, \; \forall t \in E) \tag{13}$$

because $\varphi_D(X(t,\omega))$ is continuous from the right for fixed $\omega \in \Omega$. Furthermore, if

$$\{X(t),\ t \in T\} \text{ is time-associated}$$

then

$$\{\varphi_D(X(t)),\ t \in T\} \text{ is time-associated} \tag{14}$$

because $\varphi_D(X(t))$ is a binary non-decreasing function for fixed $t \in T$, and, as

$$\{\varphi_D(X(t)),\ t \in T_k\} \text{ is a set of associated binary random variables for } D_1, D_2 \in \delta$$

$$E[\prod_{x=1}^{k} \varphi_{D_1}(X(t_x))\,\varphi_{D_2}(X(t_x))] \geqslant E[\prod_{x=1}^{k} \varphi_{D_1}(X(t_x))]\,E[\prod_{x=1}^{k} \varphi_{D_2}(X(t_x))] \tag{15}$$

is valid.

Now, assume that with respect to a proper gsp-subdigraph system $\delta_n = \{D_{in}\}$ lower and upper reliability bounds for $R_{\hat{D}_{0i}}(\tau)$, $i \in B(\delta_n)$, are known, i.e.

$$L_i(\tau) \leqslant R_{\hat{D}_{0i}}(\tau) \leqslant U_i(\tau), \quad i \in B(\delta_n), \tag{16}$$

and define

$$L_n(\tau) = \max_{D_{in} \in \delta_n} \{L_i(\tau)\,R_{D_{in}}(\tau)\}, \tag{17}$$

$$U_n(\tau) = 1 - \prod_{D_{in} \in \delta_n} (1 - U_i(\tau)\,R_{D_{in}}(\tau)). \tag{18}$$

Theorem 2. *If $\delta_n = \{D_{in}\}$ is a proper* gsp-*subdigraph system, if $\{X(t), t \in T\}$ is a time-associated reliability performance process then*

$$R_{\hat{D}_{0n}}(\tau) \geqslant L_n(\tau).$$

Proof. In consideration of (11–18) one gets for the finite set of times T_k

$$\Pr(\varphi_{\hat{D}_{0n}}(X(t)) = 1,\ t \in T_k) = \Pr(\prod_{x=1}^{k} \varphi_{\hat{D}_{0n}}(X(t_x)) = 1) = E[\prod_{x=1}^{k} \varphi_{\hat{D}_{0n}}(X(t_x))]$$

$$= E[\prod_{x=1}^{k} \max_{D_{in} \in \delta_n} \{\varphi_{\hat{D}_{0i}}(X(t_x))\,\varphi_{D_{in}}(X(t_x))\}]$$

$$\geqslant E[\max_{D_{in} \in \delta_n} \{\prod_{x=1}^{k} \varphi_{\hat{D}_{0i}}(X(t_x))\,\varphi_{D_{in}}(X(t_x))\}]$$

$$\geqslant \max_{D_{in} \in \delta_n} \{E[\prod_{x=1}^{k} \varphi_{\hat{D}_{0i}}(X(t_x))\varphi_{D_{in}}(X(t_x))]\}$$

$$\geqslant \max_{D_{in} \in \delta_n} \{E[\prod_{x=1}^{k} \varphi_{\hat{D}_{0i}}(X(t_x))]E[\prod_{x=1}^{k} \varphi_{D_{in}}(X(t_x))]\}$$

$$= \max_{D_{in} \in \delta_n} \{\Pr(\varphi_{\hat{D}_{0i}}(X(t))=1,\ t \in T_k)\Pr(\varphi_{D_{in}}(X(t))=1,\ t \in T_k)\},$$

where after application of Theorem 1 the first inequality follows from Lemma 1 (i), the second inequality from Jensen's inequality because of the convexity of the max-operator, and the third inequality from (15). Applying (12), (13) and (16), (17) gives

$$R_{\hat{D}_{0n}}(\tau) \geqslant \max_{D_{in} \in \delta_n} \{R_{\hat{D}_{0i}}(\tau)R_{D_{in}}(\tau)\} \geqslant \max_{D_{in} \in \delta_n} \{L_i(\tau)R_{D_{in}}(\tau)\} = L_n(\tau). \quad \square$$

Theorem 3. *If $\delta_n = \{D_{in}\}$ is a proper* gsp-*subdigraph system, if $\{X(t), t \in T\}$ is a time-associated reliability performance process, if the component processes $\{(X_a(t), a \in A(\delta_n)),\ t \in T\}$, $\{(X_a(t),\ a \in \overline{A(\delta_n)}),\ t \in T\}$ are independent then*

(i) $L_n(\tau) \leqslant U_n(\tau)$,

(ii) $R_{\hat{D}_{0n}}(\tau) \leqslant U_n(\tau)$,

if, additionally, $\hat{D}_{0i}$, $i \in B(\delta_n)$, $D_{in} \in \delta_n$ have lives in $T(\tau)$.

Proof.

(i) In consideration of (11–18) one gets for the finite set of times T_k

$$\max_{D_{in} \in \delta_n} \{\Pr(\varphi_{\hat{D}_{0i}}(X(t))=1,\ t \in T_k)\Pr(\varphi_{D_{in}}(X(t))=1,\ t \in T_k)\}$$

$$\leqslant E[\max_{D_{in} \in \delta_n} \{\prod_{x=1}^{k} \varphi_{\hat{D}_{0i}}(X(t_x))\varphi_{D_{in}}(X(t_x))\}]$$

$$\leqslant 1 - E[\prod_{D_{in} \in \delta_n} (1 - \prod_{x=1}^{k} \varphi_{\hat{D}_{0i}}(X(t_x))\varphi_{D_{in}}(X(t_x)))]$$

$$\leqslant 1 - \prod_{D_{in} \in \delta_n} (1 - E[\prod_{x=1}^{k} \varphi_{\hat{D}_{0i}}(X(t_x))\varphi_{D_{in}}(X(t_x))])$$

$$= 1 - \prod_{D_{in} \in \delta_n} (1 - \Pr(\varphi_{\hat{D}_{0i}}(X(t)),\ t \in T_k)\Pr(\varphi_{D_{in}}(X(t))=1,\ t \in T_k)),$$

where the first inequality follows from arguments used in the proof of Theorem 2, the second inequality from Lemma 1 (ii), the third inequality from (15) (as

$$\{1-\prod_{x=1}^{k}\varphi_{\hat{D}_{0i}}(X(t_x))\,\varphi_{D_{in}}(X(t_x)),\ D_{in}\in\delta_n\}$$

is a set of associated binary random variables) and the last equality from the independence assumption of the component processes. Applying (12), (13) and (16), (17), (18) gives the result.

(ii) Similarly,

$$\Pr(\varphi_{\hat{D}_{0n}}(X(t))=1,\ t\in T_k)=E[\prod_{x=1}^{k}\max_{D_{in}\in\delta_n}\{\varphi_{\hat{D}_{0i}}(X(t_x))\,\varphi_{D_{in}}(X(t_x))\}]$$

$$=1-E[\prod_{D_{in}\in\delta_n}(1-\prod_{x=1}^{k}\varphi_{\hat{D}_{0i}}(X(t_x))\,\varphi_{D_{in}}(X(t_x)))]$$

$$\leqslant 1-\prod_{D_{in}\in\delta_n}(1-\Pr(\varphi_{\hat{D}_{0i}}(X(t))=1,\ t\in T_k)\Pr(\varphi_{D_{in}}(X(t))=1,\ t\in T_k)),$$

where the first equality follows from Theorem 1, the second equality follows from Lemmas 1 (iii) and 2 because the gsp-subdigraphs have lives in $T(\tau)$ (where $t_1<\ldots<t_k$ is assumed for the times of T_k) and the last inequality, again, from (15) and the independence assumption of the component processes. Applying (12), (13) and (16), (18) gives

$$R_{\hat{D}_{0n}}(\tau)\leqslant 1-\prod_{D_{in}\in\delta_n}(1-R_{\hat{D}_{0i}}(\tau)R_{D_{in}}(\tau))\leqslant 1-\prod_{D_{in}\in\delta_n}(1-U_i(\tau)R_{D_{in}}(\tau))$$
$$=U_n(\tau).\quad\square$$

Of course, an interesting question is whether it is possible to get improved bounds by changing from one proper gsp-subdigraph system δ_n^* to another δ_n^{**}. The answer is given by the following.

Theorem 4. *If* $\delta_n^*=\{D_{i^*n}\}$, $\delta_n^{**}=\{D_{i^{**}n}\}$ *are proper gsp-subdigraph systems with* $\delta_n^*\subset\delta_n^{**}$, *if* $\{X(t),t\in T\}$ *is a time-associated reliability performance process then*

$$\text{(i)}\ L_{i^{**}}(\tau)R_{\hat{D}_{i^{**}i^*}}(\tau)\geqslant L_{i^*}(\tau),\qquad \begin{array}{l} i^{**}\in B(\delta_n^{**}),\\ i^*\in B(\delta_n^*)\cap N(D_{i^{**}n}),\end{array}$$
$$\Rightarrow\ L_n^{\delta_n^{**}}(\tau)\geqslant L_n^{\delta_n^*}(\tau),$$

$$\text{(ii)}\quad U_{i^{**}}(\tau)\, R_{\hat{D}_{i^{**}i^*}}(\tau) \leqslant U_{i^*}(\tau), \qquad \begin{matrix} i^{**} \in B(\delta_n^{**}), \\ i^* \in B(\delta_n^*) \cap N(D_{i^{**}n}), \end{matrix}$$

$$\Rightarrow\quad U_n^{\delta_n^{**}}(\tau) \leqslant U_n^{\delta_n^*}(\tau),$$

if, additionally, $\hat{D}_{i^{**}i^*}$, $i^{**} \in B(\delta_n^{**})$, $i^* \in B(\delta_n^*) \cap N(D_{i^{**}n})$, $D_{i^*n} \in \delta_n^*$, *have lives in* $T(\tau)$, *and, if the component processes* $\{(X_a(t), a \in \tilde{A}), t \in T\}$, $\{(X_a(t), a \in A(\delta_n^*)), t \in T\}$ *with* $A(\delta_n^*) \cup \tilde{A} = A(\delta_n^{**})$, $A(\delta_n^*) \cap \tilde{A} = 0$ *are independent.*

Here, to include the dependence of the lower (upper) reliability bounds from the chosen proper gsp-subdigraph system, the notation for the bounds is slightly different from (17), (18).

Proof. First, notice that for δ_n^*, δ_n^{**} with $\delta_n^* \subset \delta_n^{**}$ (see (10))

$$\delta_n^*/D_{i^{**}n} = \{D_{i^*n},\ D_{i^*n} \in \delta_n^* \cap D_{i^{**}n}\}$$

is a proper gsp-subdigraph system (with respect to the gsp-digraph $D_{i^{**}n}$) and Theorem 1 gives

$$\begin{aligned} \varphi_{D_{i^{**}n}} &= \max_{D_{i^*n} \in \delta_n^*/D_{i^{**}n}} \{\varphi_{\hat{D}_{i^{**}i^*}}\, \varphi_{D_{i^*n}}\} \\ &= 1 - \prod_{D_{i^*n} \in \delta_n^*/D_{i^{**}n}} (1 - \varphi_{\hat{D}_{i^{**}i^*}}\, \varphi_{D_{i^*n}}). \end{aligned} \tag{19}$$

(i) Although this is the more relevant statement of Theorem 4 the proof follows directly as a consequence of replacing $\varphi_{D_{i^{**}n}}$ by (19) and is omitted.

(ii) One has

$$\begin{aligned} & U_{i^{**}}(\tau) \prod_{x=1}^{k} \max_{D_{i^*n} \in \delta_n^*/D_{i^{**}n}} \{\varphi_{\hat{D}_{i^{**}i^*}}(X(t_x))\, \varphi_{D_{i^*n}}(X(t_x))\} \\ &= \max_{D_{i^*n} \in \delta_n^*/D_{i^{**}n}} \Big\{ \prod_{x=1}^{k} \sqrt[k]{U_{i^{**}}(\tau)}\, \varphi_{\hat{D}_{i^{**}i^*}}(X(t_x))\, \varphi_{D_{i^*n}}(X(t_x)) \Big\} \\ &\leqslant 1 - \prod_{D_{i^*n} \in \delta_n^*/D_{i^{**}n}} \Big(1 - U_{i^{**}}(\tau) \prod_{x=1}^{k} \varphi_{\hat{D}_{i^{**}i^*}}(X(t_x))\, \varphi_{D_{i^*n}}(X(t_x))\Big), \end{aligned} \tag{20}$$

where the equality follows from Lemma 1 (iii) applied to the max-expression (because the corresponding gsp-subdigraphs have lives in $T(\tau)$) and an exchange of $U_{i^{**}}(\tau)$ (which is not affected by the max-operation) with the max- and the Π-operator and the inequality from Lemma 1 (ii).

Now

$$1-\prod_{D_{i^{**}n}\in\delta_n^{**}}(1-U_{i^{**}}(\tau)\Pr(\varphi_{D_{i^{**}n}}(X(t))=1,\ t\in T_k))$$

$$=1-\prod_{D_{i^{**}n}\in\delta_n^{**}}(1-E[U_{i^{**}}(\tau)\prod_{x=1}^{k}\max_{D_{i^*n}\in\delta_n^*/D_{i^{**}n}}\{\varphi_{\hat{D}_{i^{**}i^*}}(X(t_x))\varphi_{D_{i^*n}}(X(t_x))\}])$$

$$\leqslant 1-\prod_{D_{i^{**}n}\in\delta_n^{**}}\ \prod_{D_{i^*n}\in\delta_n^*/D_{i^{**}n}}(1-U_{i^{**}}(\tau)E[\prod_{x=1}^{k}\varphi_{\hat{D}_{i^{**}i^*}}(X(t_x))\varphi_{D_{i^*n}}(X(t_x))])$$

$$=1-\prod_{D_{i^*n}\in\delta_n^*}(1-U_{i^{**}}(\tau)\Pr(\varphi_{\hat{D}_{i^{**}i^*}}(X(t))=1,\ t\in T_k)\Pr(\varphi_{D_{i^*n}}(X(t))=1,\ t\in T_k)),$$

where the first equality follows from (19), the inequality from (20) and an association property of the form

$$\{Z_1,\dots,Z_r\}\ \text{non-negative associated} \Rightarrow E\prod_{i=1}^{r}Z_i\geqslant\prod_{i=1}^{r}EZ_i$$

(which is similar to (15) and can be proved along the lines of the induction argument used in the proof of (3.1), Chapter 2 in Barlow and Proschan [2]) and the last equality from the independence assumption. Applying (12), (13) and (16), (18) gives the result. □

Of course, the decision which of the proper gsp-subdigraph systems δ_n^*, δ_n^{**} with $\delta_n^*\subset\delta_n^{**}$ one should choose will depend on the fact, how good the given bounds L_{i^*}, U_{i^*}, $i^*\in B(\delta_n^*)$, $L_{i^{**}}$, $U_{i^{**}}$, $i^{**}\in B(\delta_n^{**})$, respectively, are as well as on the knowledge about and the difficulties for the possibilities of the determination of the reliability of the gsp-subdigraphs of the proper systems.

Instead of formulating further conditions for such an optimal choice the next section shows that the just developed approach allows remarkable improvements even in an example of simplest form. From the preceding theorems one gets the following.

Procedure

Step n. Choose $n\in N(\hat{D}_{0m})$, $1\leqslant n\leqslant m$, and a proper gsp-subdigraph system $\delta_n=\{D_{in}\}$ for which lower (upper) reliability bounds $L_i(\tau)$ $(U_i(\tau))$, $i\in B(\delta_n)$, are known (with $L_0(\tau)\equiv U_0(\tau)\equiv 1$). Determine $L_n^{\delta_n}(\tau)$ $(U_n^{\delta_n}(\tau))$.

Use these bounds to increase n in a suitable way (remember the level assignment of the nodes) until m is reached.

5. Example

To illustrate the theoretical results the following example is given to show that bounds additional to the minimal cut lower bound of Esary and Proschan could be of interest for reliability estimation. In the gsp-digraph of Fig. 1 the numbers attached to the nodes give a possible level assignment. The cross-connection arc (2, 3) from 2 to 3 hinders straightforward series-parallel reduction.

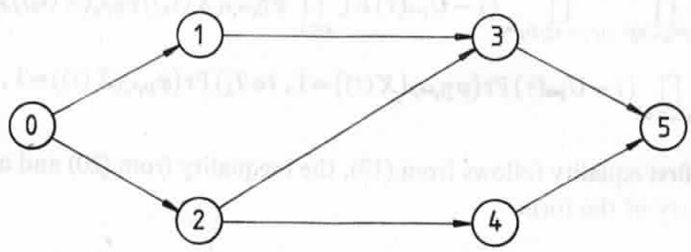

FIG. 1.

The arcs are assumed to have lives in $T(\tau)$ with lifetimes independently identically exponential-distributed with parameter λ, i.e.

$$\Pr(X_a(t)=1,\ \forall t\in T(\tau))=\Pr(\mathrm{Life}_a>\tau)=e^{-\lambda\tau}(=\Pr(X_a(\tau)=1)), \quad (21)$$

where Life_a is the random variable describing the lifetime of arc a.

As there are:

three paths

$$P^1_{05}=(\{0,1,3,5\},\{(0,1),(1,3),(3,5)\})$$
$$P^2_{05}=(\{0,2,3,5\},\{(0,2),(2,3),(3,5)\})$$
$$P^3_{05}=(\{0,2,4,5\},\{(0,2),(2,4),(4,5)\})$$

and nine minimal cuts

$$C^1_{05}=\{(0,1),(0,2)\},\quad C^6_{05}=\{(0,1),(2,3),(2,4)\}$$
$$C^2_{05}=\{(0,2),(1,3)\},\quad C^7_{05}=\{(1,3),(2,3),(2,4)\}$$
$$C^3_{05}=\{(0,2),(3,5)\},\quad C^8_{05}=\{(0,1),(2,3),(4,5)\}$$
$$C^4_{05}=\{(2,4),(3,5)\},\quad C^9_{05}=\{(1,3),(2,3),(4,5)\}$$
$$C^5_{05}=\{(3,5),(4,5)\}$$

one easily gets from (21)

$$R_{P^i_{05}}(\tau)=e^{-3\lambda\tau},\ i\in\{1,\dots,3\},\ R_{C^i_{05}}(\tau)=\begin{cases}1-(1-e^{-\lambda\tau})^2, & i\in\{1,\dots,5\},\\ 1-(1-e^{-\lambda\tau})^3, & i\in\{6,\dots,9\}.\end{cases} \tag{22}$$

The easiest bounds using path representation are

$$\begin{aligned}L^1_5(\tau)&=\max_{P_{05}\in P(\hat{D}_{05})}\{R_{P_{05}}(\tau)\}=e^{-3\lambda\tau},\\ U^1_5(\tau)&=1-\prod_{P_{05}\in P(\hat{D}_{05})}(1-R_{P_{05}}(\tau))=1-(1-e^{-3\lambda\tau})^3.\end{aligned} \tag{23}$$

For the proper gsp-subdigraph system $\delta_5=\{D_{25},\hat{D}_{35}\}$

$$\begin{aligned}&\text{with } D_{25}=(\{2,4,5\},\{(2,4),(4,5)\}), && R_{D_{25}}(\tau)=e^{-2\lambda\tau},\\ &\text{and } \hat{D}_{35}=(\{3,5\},\{(3,5)\}), && R_{\hat{D}_{35}}(\tau)=e^{-\lambda\tau},\end{aligned}$$

no lower (upper) bounds for $\hat{D}_{0i}$, $i\in B(\delta_5)$, are needed because the exact reliability

$$\begin{aligned}&R_{\hat{D}_{02}}(\tau)=e^{-\lambda\tau} && \text{for } \hat{D}_{02}=(\{0,2\},\{(0,2)\}),\\ &R_{\hat{D}_{03}}(\tau)=1-(1-e^{-2\lambda\tau})^2 && \text{for } \hat{D}_{03}=(\{0,1,2,3\},\{(0,1),(0,2),\\ & && (1,3),(2,3)\}),\end{aligned}$$

is easily obtainable, the corresponding bounds (see (17), (18)) are

$$\begin{aligned}L^2_5(\tau)&=(1-(1-e^{-2\lambda\tau})^2)e^{-\lambda\tau},\\ U^2_5(\tau)&=1-(1-e^{-\lambda\tau}e^{-2\lambda\tau})(1-(1-(1-e^{-2\lambda\tau})^2)e^{-\lambda\tau}).\end{aligned} \tag{24}$$

For the minimal cut lower bound of Esary and Proschan one gets from (22)

$$EP_{\hat{D}_{05}}(\tau)=(1-(1-e^{-\lambda\tau})^2)^5(1-(1-e^{-\lambda\tau})^3)^4. \tag{25}$$

The bounds (23), (24), (25) are given in Table 1 together with the exact reliability value (denoted by R) for different values of λ and τ. The results indicate that for high reliabilities the EP-bound can be better but that the suggested decomposition of gsp-digraphs with respect to proper gsp-subdigraph systems (for which improvements according to Theorem 4 are possible) becomes more and more interesting for increasing time.

Table 1

L_i, U_i, EP reliability bounds for R with $L_1 \leq L_2 \leq R \leq U_2 \leq U_1$ and $EP \leq R$

$\lambda=$	$\tau=$	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	L_1	0.74082	0.54881	0.40657	0.30119	0.22313	0.16530	0.12246	0.09072	0.06721	0.04979
	L_2	0.87511	0.72974	0.59001	0.46705	0.36418	0.28081	0.21472	0.16312	0.12330	0.09284
	U_2	0.96763	0.87806	0.75670	0.62757	0.50605	0.39969	0.31088	0.23904	0.18222	0.13800
0.10	U_1	0.98259	0.90815	0.79102	0.65875	0.53114	0.41844	0.32422	0.24821	0.18837	0.14205
	R	0.95717	0.85608	0.73061	0.60300	0.48563	0.38399	0.29943	0.23099	0.17672	0.13432
	EP	0.95224	0.82617	0.65840	0.48613	0.33540	0.21796	0.13436	0.07907	0.04468	0.02435
	L_1	0.54881	0.30119	0.16530	0.09072	0.04979	0.02732	0.01500	0.00823	0.00452	0.00248
	L_2	0.72974	0.46705	0.28081	0.16312	0.09284	0.05217	0.02908	0.01612	0.00891	0.00491
	U_2	0.87806	0.62757	0.39969	0.23904	0.13800	0.07807	0.04364	0.02422	0.01339	0.00738
0.20	U_1	0.90815	0.65875	0.41844	0.24821	0.14205	0.07975	0.04432	0.02449	0.01349	0.00742
	R	0.85608	0.60300	0.38399	0.23099	0.13432	0.07649	0.04299	0.02396	0.01329	0.00734
	EP	0.82617	0.48613	0.21796	0.07907	0.02435	0.00660	0.00162	0.00037	0.00008	0.00002
	L_1	0.40657	0.16530	0.06721	0.02732	0.01111	0.00452	0.00184	0.00075	0.00030	0.00012
	L_2	0.59001	0.28081	0.12330	0.05217	0.02166	0.00891	0.00365	0.00149	0.00061	0.00025
	U_2	0.75670	0.39969	0.18222	0.07807	0.03253	0.01339	0.00547	0.00223	0.00091	0.00037
0.30	U_1	0.79102	0.41844	0.18837	0.07975	0.03296	0.01349	0.00550	0.00224	0.00091	0.00037
	R	0.73061	0.38399	0.17672	0.07649	0.03212	0.01329	0.00545	0.00223	0.00091	0.00037
	EP	0.65840	0.21796	0.04468	0.00660	0.00078	0.00008	0.00001	0.00000	0.00000	0.00000
	L_1	0.30119	0.09072	0.02732	0.00823	0.00248	0.00075	0.00022	0.00007	0.00002	0.00001
	L_2	0.46705	0.16312	0.05217	0.01612	0.00491	0.00149	0.00045	0.00014	0.00004	0.00001
	U_2	0.62757	0.23904	0.07807	0.02422	0.00738	0.00223	0.00067	0.00020	0.00006	0.00002
0.40	U_1	0.65875	0.24821	0.07975	0.02449	0.00742	0.00224	0.00067	0.00020	0.00006	0.00002
	R	0.60300	0.23099	0.07649	0.02396	0.00734	0.00223	0.00067	0.00020	0.00006	0.00002
	EP	0.48613	0.07907	0.00660	0.00037	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000

	L_1	0.22313	0.04979	0.01111	0.00248	0.00055	0.00012	0.00003	0.00001	0.00000	0.00000
	L_2	0.36418	0.09284	0.02166	0.00491	0.00110	0.00025	0.00006	0.00001	0.00000	0.00000
	U_2	0.50605	0.13800	0.03253	0.00738	0.00165	0.00037	0.00008	0.00002	0.00000	0.00000
0.50	U_1	0.53114	0.14205	0.03296	0.00742	0.00166	0.00037	0.00008	0.00002	0.00000	0.00000
	R	0.48563	0.13432	0.03212	0.00734	0.00165	0.00037	0.00008	0.00002	0.00000	0.00000
	EP	0.33540	0.02435	0.00078	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	L_1	0.10540	0.01111	0.00117	0.00012	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000
	L_2	0.18728	0.02166	0.00233	0.00025	0.00003	0.00000	0.00000	0.00000	0.00000	0.00000
	U_2	0.27294	0.03253	0.00350	0.00037	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000
0.75	U_1	0.28404	0.03296	0.00351	0.00037	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000
	R	0.26330	0.03212	0.00349	0.00037	0.00004	0.00000	0.00000	0.00000	0.00000	0.00000
	EP	0.10364	0.00078	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	L_1	0.04979	0.00248	0.00012	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	L_2	0.09284	0.00491	0.00025	0.00001	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	U_2	0.13800	0.00738	0.00037	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
1.00	U_1	0.14205	0.00742	0.00037	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	R	0.13432	0.00734	0.00037	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	EP	0.02435	0.00002	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

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