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R-SATISFYING MULTI-COMMODITY FLOWS AND CAPACITY-ALTERATIONS OF NETWORKS*1)

Summary. It is shown that the problem of altering the capacity of a multi-commodity network under the restriction that multi-commodity flows with prescribed flow values have to exist in the net (which leads to large linear programs) can be solved by using known network algorithms as subprograms.

Introduction

The problem of increasing the capacity of a (one commodity) network (to make it feasible for a (one commodity) flow with prescribed flow-value) has been treated by several researchers (see FULKERSON [4], HAMMER [6], HESS [7], HU [8], PRICE [10]). In the two-commodity case GAUL [5] has considered conditions under which this problem can be reduced to the determination of capacity--increasings of one-commodity networks. In the general case (with multi-commodity flows taken into consideration) this paper describes sol ution procedures using the column generating technique of FORD/FULKERSON [3].

Existence of r-satisfying k-commodity flows

Let N be a finite set with $\# N=n$ and $A \subset N\times N \setminus \cup\{(i, i) \mid i\in N\}$. If the simple directed finite graph (N, A) is connected, it is called a network, and if it, additionally, is complete and symmetric it is denoted by (N, c) where $c: A\to R_+$ describes the capacity c_{ij} of the arc $a_{ij}\in A$ connecting the nodes $i, j\in N$. Then we have $\# A=n(n-1)$. The arcs a_{ij} with $c_{ij}=0$ may be considered to represent the non-existing arcs.

$$(x^1, \ldots, x^k) \text{ with } x^s: A\to R,\ s, \hat{s}\in N\ (s=1, \ldots, k) \text{ and}$$

$$\sum_j \overset{s}{x}_{ij}-\sum \overset{s}{x}_{ji}=\begin{cases} v(x^s) & i=s \\ 0 & i\neq s, \hat{s} \\ -v(x^s) & i=\hat{s}\end{cases} \qquad s=1, \ldots, k \tag{1}$$

$$\sum_{s=1}^{k} \overset{s}{x}_{ij}\leq c_{ij}, \quad \overset{s}{x}_{ij}\geq 0 \qquad i, j\in N \quad (i\neq j) \tag{2}$$

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is called a *feasible k-commodity flow for* (N, c), $v(x^s)$ is called *flow-value* of x^s $v(1, \ldots, k) := \sum_{s=1}^{k} v(x^s)$ *flow-value* of $(x^1, \ldots, x^k)$, x_{ij}^s *flow of the kind s in the arc* a_{ij}. Let be $r = (r^1, \ldots, r^k)' \in R_+^k$. $(x^1, \ldots, x^k)$ is called *r-satisfying* if $v(x^s) \geqslant r^s$ $(s = 1, \ldots, k)$. The set of r-satisfying feasible k-commodity flows for (N, c) is denoted by $F(N, c)_r$, (N, c) is called *r-feasible* if $F(N, c)_r \neq \emptyset$. $y \in R_+^{n(n-1)}$ is called an *r-feasible capacity-increasing* for (N, c), if $F(N, c+y)_r \neq \emptyset$. Let p: $A \to R_+$ describe the price p_{ij} of constructing an additional unit of y_{ij}, then the problem consists in finding

$$\min \sum p_{ij} y_{ij}$$

$$\text{(3)} \qquad \sum_j x_{ij}^s - \sum_j x_{ji}^s \begin{cases} \geqslant r^s & i = s \\ = 0 & i \neq s, \hat{s} \\ \leqslant -r^s & i = \hat{s} \end{cases} \qquad \begin{matrix} s = 1, \ldots, k \\ s \neq \hat{s} \end{matrix}$$

$$\text{(4)} \qquad \sum_{s=1}^{k} x_{ij}^s - y_{ij} \leqslant c_{ij}, \quad x_{ij}^s \geqslant 0, \quad y_{ij} \geqslant 0.$$

Let $P_l(s, \hat{s})$ denote a shortest path from s to $\hat{s}$, where l: $A \to R_+$ describes the length l_{ij} of the arc a_{ij}, and $L(P_l(s, \hat{s}))$ the length of $P_l(s, \hat{s})$ $(s = 1, \ldots, k)$. Then one can state (*IRI*) [9])

$$F(N, c)_r \neq \emptyset \Leftrightarrow c' l \geqslant \sum_{s=1}^{k} r^s L(P_l(s, \hat{s})) \quad \text{for all} \quad l \in R_+^{n(n-1)}.$$

In order to see that the statement is correct one has to remember that from the FARKAS [1] — lemma one knows that (3) (4) with $y_{ij} = 0$ for all $i, j \in N$ $(i \neq j)$ is solvable if and only if for all l_{ij}, π_q^s $(i, j, q \in N, s = 1, \ldots, k)$ which fulfill

$$\text{(5)} \qquad \pi_i^s - \pi_j^s + {}_{ij} \geqslant 0$$

$$\text{(6)} \qquad \pi_s^s \leqslant 0, \quad \pi_{\hat{s}}^s \geqslant 0 \qquad s = 1, \ldots, k$$

$$\text{(7)} \qquad l_{ij} \geqslant 0 \quad \text{it follows}$$

$$\text{(8)} \qquad \sum_{a_{ij} \in A} c_{ij} l_{ij} \geqslant \sum_{s=1}^{k} r^s (\pi_{\hat{s}}^s - \pi_s^s)$$

As the rank of the (node-arc) incidence matrices corresponding to (3) is $n-1$, one can set $\pi_s^s = 0$ $(s = 1, \ldots, k)$. Then, for given $l_{ij} \geqslant 0$ (5), (6), (7) can be interpreted as the description of the situation for determining shortest paths from s to $\hat{s}$ where π_q^s is the lenght of a shortest path from s to q.

The statetment is of less practical importance as one has to compute $c' l$, $P_l(s, \hat{s})$ and $L(P_l(s, \hat{s}))$ $(s = 1, \ldots, k)$, for all given l: $A \to R_+$ to prove $F(N, c)_r \neq \emptyset$. In the next section we shall give efficient solution procedures for this problem. Other existence-theorems for multi-commodity flows are available if consideration is constrained to the two-commodity case (see GAUL [5]).

Solution procedures using the solumn generating technique Let be $\mathcal{P}=\bigcup_{s=1}^{k}\mathcal{P}^s$ and $\mathcal{P}^s=\{P_1^s, \ldots, P_{n(s)}^s\}$ the set of all paths from s to $\hat{s}$. Let x_e^s denote the amount of the flow x^s in the path P_e^s. The matrix $B=(b_{ije}^s)$ with

$$b_{ije}^s=\begin{cases}1 & a_{ij}\in P_e^s\\ 0 & \text{otherwise}\end{cases}$$

is called *arc-path incidence matrix*.

With the price-vector $p\in R_+^{n(n-1)}$ we get the following linear program

$$\min \sum_{a_{ij}\in A} p_{ij}\, y_{ij}$$

(9) $$\sum_{s=1}^{k}\sum_{e=1}^{n(s)} b_{ije}^s\, x_e^s - y_{ij}\leqslant c_{ij} \qquad i, j\in N\ (i\neq j)$$

(10) $$\sum_{e=1}^{n(s)} x_e^s \geqslant r^s \qquad s=1, \ldots, k$$

$$x_e^s\geqslant 0,\quad y_{ij}\geqslant 0$$

If we assign simplex multipliers u_{ij} to (9), v^s to (10), then because of the rules of the revised simplex method x_e^s or y_{ij}, respectively, will become basic if

(11) $$\sum_{a_{ij}\in A}(-u_{ij})\, b_{ije}^s - v^s<0 \qquad \text{for } x_e^s$$

(12) $$-u_{ij}>p_{ij} \qquad \text{for } y_{ij} \quad \text{where}$$

(13) $$(-u_{ij})\geqslant 0 \quad i, j\in N\ (i\neq j);\quad v^s\geqslant 0\ (s=1, \ldots, k)$$

should be valid all times. If $(-u_{ij})<0$ or $v^s<0$ one only has to introduce the columns of the corresponding slackvariables into the basis to guarantee (13). There is no difficulty to check (12) as the column belonging to y_{ij} is known. To check (11) it is not necessary to know the enormous number of columns corresponding to x_e^s $(e=1, \ldots, n(s);\ s=1, \ldots, k)$. Interpreting $(-u_{ij})\geqslant 0$ as length of the arc a_{ij} a shortest path algorithm may be used to search for a path $(b_{ije_s}^s \mid i, j\in N\ (i\neq j))$ satisfying (11) for commodity s. If there exists an optimum solution with $y_{ij}=0$ for $i, j\in N\ (i\neq j)$, then (N, c) was r-feasible and $F(N, c)_r\neq\emptyset$.

We, now will describe another formulation of the above problem. Let denote $y\in R_+^{n(n-1)}$ the increasing, $z\in R_+^{n(n-1)}$ the decreasing of the capacity of (N, c) and $p^y, p^z\in R_+^{n(n-1)}$ the price for doing so. We have $z\leqslant c$. Let $q^c, q^y\in R_+^{n(n-1)}$ denote the price of maintaining the capacity c or y, reprectively. Then we get the cost

$$(p^y+q^y)'\, y+p^{z'}\, z+q^{c'}\,(c-z) \text{ and the problem}$$

$$\min \sum\left((p_{ij}^y+q_{ij}^y)\, y_{ij}+(p_{ij}^z-q_{ij}^c)\, z_{ij}\right)$$

(14) $$\sum_j x^s_{ij}-\sum_j x^s_{ji}\begin{cases}\geqslant r^s & i=s\\ =0 & i\neq s,\,\hat{s}\\ \leqslant -r^s & i=\hat{s}\end{cases}\qquad \begin{matrix} s=1,\,\dots,\,k\\ s\neq\hat{s}\end{matrix}$$

(15) $$\sum_{s=1}^{k} x^s_{ij}-y_{ij}+z_{ij}\leqslant c_{ij}$$

(16) $$z_{ij}\leqslant c_{ij},\quad x^s_{ij}\geqslant 0,\quad y_{ij}\geqslant 0,\quad z_j\geqslant 0$$

$$K^s=\left\{x^s_{ij}\,\middle|\,\sum x^s_{ij}-\sum x^s_{ji}\begin{cases}\geqslant r^s & i=s\\ =0 & i\neq s,\,\hat{s},\\ \leqslant -r^s & i=\hat{s}\end{cases}\begin{matrix} s=1,\,\dots,\,k,\\ s\neq\hat{s}\end{matrix}\quad x^s_{ij}\geqslant 0,\quad \begin{matrix} i,\,j\in N\\ i\neq j\end{matrix}\right\}$$

is a convex (unbounded) polytope. Let $x^s_e\in R^{n(n-1)}_+$ $(e=1,\,\dots,\,m(s))$ the be extreme points of this convex set. Then, every $x^s\in K^s$ can be expressed by the following form

$$x^s=\sum_{e=1}^{m(s)} d^s_e\,x^s_e\qquad \sum_{d=1}^{m(s)} d^s_e\geqslant 1,\qquad d^s_e\geqslant 0$$

and (14), (15), (16) is changed into

$$\min\{(p^y+q^y)'\,y+(p^z-q^c)'\}$$

(17) $$\sum_{s=1}^{k}\sum_{e=1}^{m(s)} x^s_e\,d^s_e-y-z\leqslant c$$

(18) $$z\leqslant c$$

(19) $$\sum_{e=1}^{m(s)} d^s_e\geqslant 1$$

$$d^s_e\geqslant 0,\quad y\geqslant 0,\quad z\geqslant 0$$

Conditions for generating a new basis are

(20) $$(-u)'x^s_e-v^s<0.\qquad \text{for } d^s_e$$

(21) $$-u_{ij}>p^y_{ij}+q^y_{ij}\qquad \text{for } y_{ij}$$

(12) $$p^z_{ij}-q^c_{ij}-u_{ij}-w_{ij}<0\quad \text{for } z_{ij}$$

if we assign simplex multipliers $u,\ w\in R^{n(n-1)}$ to (17), (18), $v^s\in R$ $(s=1,\dots,k)$ to (19), respectively, where

(23) $$u\leqslant O,\quad w\leqslant O,\quad v^s\geqslant O\quad (s=1,\,\dots,\,k)$$

should be valid at all times. To quarantee (23), one only has to introduce into the basis the columns of those slack-variables for which the corresponding simplex multipliers violate (23). There are no dificulties in cheching (21), (22). For (20) the most improving column can be found by solving the k suprograms

$$\min\sum_{a_{ij}\in A}(-u_{ij})\,x^s_{ij}$$

$$\sum_j x_{ij}^s - \sum_j x_{ji}^s \begin{cases} \geq r^s & i=s \\ =0 & i \neq s, \hat{s}, \quad x_{ij}^s \geq 0 \\ \leq -r^s & i=\hat{s} \end{cases} \tag{24}$$

Each of these programs (24) is a one- commodity flow problem in an uncapacitated net and can be solved by interpreting $(-u_{ij}) \geq 0$ as the length of the arc a_{ij}, determining a shortest part from s to $\hat{s}$ and sending r^s along this path. Is should be stated that the formulations (9), (10) and (17), (18), (19) are equlvalent except for the additional decision z (see FORD/FULKERSON [2] where the equivalence of flows in node-arc- and arc-path-form is proved).

If there exists an optimum solution with $y_{ij}=0$ for $i, j \in N$ $(i \neq j)$ then $(N, c-z)$ is r-feasible and thus $F(N, c)_r \neq \emptyset$.

There is still anothe possibility of describing the above problem. Let $\mathcal{M}_N \subset R_+^{n(n-1)}$ be the set of all r-feasible networks with reference to N (for $\tilde{c}=(\tilde{c}_1, \ldots, \tilde{c}_{n(n-1)})' \in \mathcal{M}_N$ tke capacity of the i-th arc is denoted by $\tilde{c}_i$). $\mathcal{M}_N$ is a convex (unbounded) polytope. Let c^e $(e=1, \ldots, M)$ be the extreme points of $\mathcal{M}_N$. Then we have

$$\mathcal{M}_N=\left\{\tilde{c} \mid \tilde{c}=\sum_{e=1}^M d_e c^n, \quad \sum_{e=1}^M d_e \geq 1, \quad d_e \geq 0\right\}$$ and the problem

$$\min (p^y+q^y)' y+(p^z-q^c)' z$$

$$\sum_{e=1}^M d_e c^e - y + z \leq c \tag{25}$$ $$z \leq c \tag{26}$$

$$\sum_{e=1}^M d_e \geq 1 \qquad d_e \geq O, \quad y \geq O, \quad z \geq O \tag{27}$$

If simplex multipliers $u, w \in R_+^{n(n-1)}$, $v \in R$ are assigned to (25), (26), (27), the conditions for a basis change are

(28) $(-u)' c^e - v < 0$ for d_e (29) $-u_{ij} > p_{ij}^y + q_{ij}^y$ for y_{ij}

(30) $p_{ij}^z - q_{ij}^c - u_{ij} - w_{ij} < 0$ for z_{ij}. Again

$$u \leq O, \quad w \leq O, \quad v \geq O \tag{31}$$

should be valid and can be guaranteed by using corresponding slack-rariables.

(28), (29) are easy to check, (30) needs the determination of a cheapest r-feasible network using $(-u_{ij}) \geq O$ as price for constructing an additional unit of capacity for arc a_{ij}. To construct a cheapest r-feasible net one only needs to know N and the costs $(-u_{ij}) \geq O$, $i, j \in N$ $(i \neq j)$. After the determination of shortest paths $P(s, \hat{s})$, $s=1, \ldots, k$, each arc $a_{ij} \in \bigcup_{s=1}^k P(s, \hat{s})$ will get the capacity

$$\hat{c}_{ij}=\sum_{s=1}^k h_{ij}^s r^s \quad \text{with} \quad h_{ij}^s=\begin{cases}1 & \text{if } a_{ij} \in P(s, \hat{s}) \\ 0 & \text{otherwice.}\end{cases}$$

It is easy to see that $F(N, \hat{c})_r \neq \emptyset$ and that $(N, \hat{c})$ is a cheapest net with this property.

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4.37. Gherardelli, Francesco: Two problems*

4.37.1. Problem. Let V be an algebraic manifold of complex dimension n. Let G_p = group of algebraic cycles of V which are algebraically zero. It is known that there is a natural map W of G_p into the complex torus

$H^{2n-2p-1}$ $(V, R/\text{Im}\, H^{2n-2p-1}$ (X, Z) (Weil: *Picard Varieties* Journal of Mat 481) and the image of W is an abelian manifold $\mathcal{T}_p$ (P. Liebermann, Journal of Math. **72** Griffiths Journal of Math. 1971—72.).

To show that $\mathcal{T}_{(p)}$ *and* $\mathcal{T}_{(n-p-1)}$ *are dual as abelian manifolds.* (This is true for $p=0$ (classical) and it has been shown also true for $n=2p+1$ by Griffiths-Bloch (Annals of Math. (1973)).

Other bibliography: a paper by Stephen Bloch in Annals of Mathematics 1974.

4.37.2. Problem. Let K be an algebraic number field of degree $s+2t$ (s = number of fields conjugate to K and real, $2t$ = number of conjugate fields which are complex but not real.)

$K \otimes R \simeq R^s \times C^t \subset C^{t+s}$ embed R^s in C^s as real part. Let A = ring of integers of K; *find cases in which* C^{t+s}/A *is an algebraic manifold* (non compact if $s=0$) (true, if $s=0$, K imaginary quadratic extension of a totally real field; $s=t$, K cubic non totally real extension of a totally real field.)

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